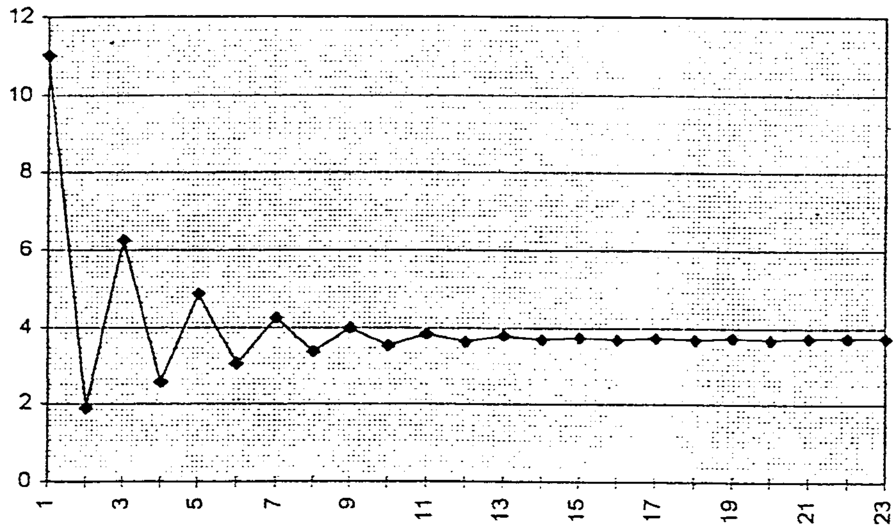
V. SELEACU I. BĂLĂCENOIU

SMARANDACHE NOTIONS

(book series)

Vol. 10



American Research Press

1999

FOREWARD

A collection of papers concerning Smarandache type functions, numbers, sequences, integer algorithms, paradoxes, experimental geometries, algebraic structures, neutrosophic probability, set, and logic, etc. is published this year.

V. Seleacu & I. Balacenoiu
Department of Mathematics
University of Craiova, Romania

Smarandache Factors and Reverse Factors

Micha Fleuren

November 1998

Abstract

This document will describe the current status on the search for factors of Smarandache consecutive numbers and their reverse. A complete list up to index 200 will be given, with all known factors. Smarandache numbers are the concatenation of the natural numbers from one up to the given index, and reverse Smarandache numbers are the concatenation of the natural numbers from the given index down to 1.

1 Introduction

As a followup to Ralf Stephan's article in this journal [St], I decided to extend his factorizations to index 200. The Smarandache consecutive sequence, as well as their reverse is described in [Sm]. In this article Sm11 denotes 1234567891011 for example, and Rsm11 denotes 1110987654321.

Most of the factors that have been found by me and others, have been found by using the elliptic curve method (ECM) [Le], some have been found using the Multiple Polynomial Quadratic Sieve (MPQS) [Si].

All factors and remaining cofactors have been proven prime or composite by means of Elliptic Curve Primality Proving (ECPP) [At], or the Adleman-Pomerance-Rumely test [Ad], which has been simplified in 1984 by Cohen and Lenstra [Co].

2 Software used

The main factoring program used has been `GMP-ECM` by Paul Zimmermann [Zi, Le, Gr]. The first small factors were filtered out quickly by `ECMX`, a program of the `UBASIC` package [Ki, Le].

The factors which were probably prime were then tested with François Morain's `ECPP` [Mo, At]. Some factors have been proven prime by use of `APRT-CLE` [Ad] from the `UBASIC` package [Ki].

All these fine pieces of software are freely available from the internet. The appropriate adresses are enlisted in the references.

3 Progress of calculations

All numbers have been factored using GMP-ECM up to 20 digits. First 25 runs with $B1 = 2000$ were run, and if the factorization wasn't complete, 90 runs with $B1 = 11000$ were run.

Work is in progress to extend this to 25 digits. Some factors have already been tried to 25 digits (300 curves with $B1 = 50000$). For more detail on the progress check the following URL:
`http://www.sci.kun.nl/sigma/Persoonlijk/michaf/ecm/ecmtries.html`
Currently the lowest not-completely factored numbers are Sm63 and Rsm59.

4 Factorization results

The lists presented here are an up to date representation of the factors known so far. When more factors are found they will be added to the list, which can be found on the internet at the following URL:
`http://www.sci.kun.nl/sigma/Persoonlijk/michaf/ecm/`

Most of the factors up to Sm80 and Rsm80 should be creditted to Ralf Stephan. (unless otherwise stated). All contributors, together with their email-adresses can be found in tables 1 and 3.

A '*' denotes an uncomplete factorization, pxx denotes a prime of xx digits and cxx denotes a composite number of xx digits.

4.1 Smarandache Factors

Contributors of Smarandache factors		
RB	Robert Backstrom	`bobb@atinet.com.au`
TC	Tim Charron	`tcharron@interlog.com`
BD	Bruce Dodson	`bad0@lehigh.edu`
MF	Micha Fleuren	`michaf@sci.kun.nl`

AM	Allan MacLeod	MACL-MS0@wpmail.paisley.ac.uk
RS	Ralf Stephan	stephan@tmt.de
EW	Egon Willighagen	egonw@sci.kun.nl
PZ	Paul Zimmermann	zimmerma@loria.fr (LORIA, Nancy, France)

Table 1: Contributors of Smarandache factors

n	Factors of Sm(n)
2	$2^2 \cdot 3$
3	$3 \cdot 41$
4	$2 \cdot 617$
5	$3 \cdot 5 \cdot 823$
6	$2^6 \cdot 3 \cdot 643$
7	$127 \cdot 9721$
8	$2 \cdot 3^2 \cdot 47 \cdot 14593$
9	$3^2 \cdot 3607 \cdot 3803$
10	$2 \cdot 5 \cdot 1234567891$
11	$3 \cdot 7 \cdot 13 \cdot 67 \cdot 107 \cdot 630803$
12	$2^3 \cdot 3 \cdot 2437 \cdot 2110805449$
13	$113 \cdot 125693 \cdot 869211457$
14	$2 \cdot 3$
	$p18 : 205761315168520219$
15	$3 \cdot 5$
	$p19 : 8230452606740808761$
16	2^2
	$p10 : 2507191691$
	$p13 : 1231026625769$
17	$3^2 \cdot 47 \cdot 4993$
	$p18 : 584538396786764503$
18	$2 \cdot 3^2 \cdot 97 \cdot 88241$
	$p18 : 801309546900123763$
19	$13 \cdot 43 \cdot 79 \cdot 281 \cdot 1193$
	$p18 : 833929457045867563$
20	$2^5 \cdot 3 \cdot 5 \cdot 323339 \cdot 3347983$
	$p16 : 2375923237887317$
21	$3 \cdot 17 \cdot 37 \cdot 43 \cdot 103 \cdot 131 \cdot 140453$
	$p18 : 802851238177109689$
22	$2 \cdot 7 \cdot 1427 \cdot 3169 \cdot 85829$
	continued...

n	Factors of $\mathrm{Sm}(n)$
	$p22$: 2271991367799686681549
23	$3 \cdot 41 \cdot 769$
	$p32$: 13052194181136110820214375991629
24	$2^2 \cdot 3 \cdot 7$
	$p18$: 978770977394515241
	$p19$: 1501601205715706321
25	$5^2 \cdot 15461$
	$p11$: 31309647077
	$p25$: 1020138683879280489689401
26	$2 \cdot 3^4 \cdot 21347 \cdot 2345807$
	$p12$: 982658598563
	$p18$: 154870313069150249
27	$3^3 \cdot 19^2 \cdot 4547 \cdot 68891$
	$p32$: 40434918154163992944412000742833
28	$2^3 \cdot 47 \cdot 409$
	$p15$: 416603295903037
	$p27$: 192699737522238137890605091
29	$3 \cdot 859$
	$p20$: 24526282862310130729
	$p26$: 19532994432886141889218213
30	$2 \cdot 3 \cdot 5 \cdot 13 \cdot 49269439$
	$p18$: 370677592383442753
	$p23$: 17333107067824345178861
31	29
	$p10$: 2597152967
	$p42$: 163915283880121143989433769727058554332117
32	$2^2 \cdot 3 \cdot 7$
	$p23$: 45068391478912519182079
	$p30$: 326109637274901966196516045637
33	$3 \cdot 23 \cdot 269 \cdot 7547$
	$p18$: 116620853190351161
	$p31$: 7557237004029029700530634132859
34	2
	$p50$: 61728394550556065707580859096010611162126313641465 15661667
35	$3^2 \cdot 5 \cdot 139 \cdot 151 \cdot 64279903$
	$p10$: 4462548227
	$p37$: 4556722495899317991381926119681186927
36	$2^4 \cdot 3^2 \cdot 103 \cdot 211$
	$p56$
	continued...

n	Factors of $\mathrm{Sm}(n)$
37	$71 \cdot 12379 \cdot 4616929$
	$p52$
38	$2 \cdot 3$
	$p23$: 86893956354189878775643
	$p43$: 2367958875411463048104007458352976869124861
39	$3 \cdot 67 \cdot 311 \cdot 1039$
	$p25$: 6216157781332031799688469
	$p36$: 305788363093026251381516836994235539
40	$2^2 \cdot 5 \cdot 3169 \cdot 60757 \cdot 579779$
	$p10$: 4362289433
	$p20$: 79501124416220680469
	$p26$: 15944694111943672435829023
41	$3 \cdot 487 \cdot 493127 \cdot 32002651$
	$p56$
42	$2 \cdot 3 \cdot 127 \cdot 421$
	$p11$: 22555732187
	$p25$: 4562371492227327125110177
	$p34$: 3739644646350764691998599898592229
43	$7 \cdot 17 \cdot 449$
	$p72$
44	$2^3 \cdot 3^2$
	$p26$: 12797571009458074720816277
	$p52$
45	$3^2 \cdot 5 \cdot 7 \cdot 41 \cdot 727 \cdot 1291$
	$p13$: 2634831682519
	$p18$: 379655178169650473
	$p41$: 10181639342830457495311038751840866580037
46	$2 \cdot 31 \cdot 103 \cdot 270408101$
	$p18$: 374332796208406291
	$p25$: 3890951821355123413169209
	$p28$: 4908543378923330485082351119
47	$3 \cdot 4813 \cdot 679751$
	$p22$: 4626659581180187993501
	$p53$
48	$2^2 \cdot 3 \cdot 179 \cdot 1493 \cdot 1894439$
	$p29$: 15771940624188426710323588657
	$p46$: 1288413105003100659990273192963354903752853409
49	$23 \cdot 109 \cdot 3251653$
	$p10$: 2191196713

continued...

n	Factors of Sm(n)
	$p23$: 53481597817014258108937
	$p47$: 12923219128084505550382930974691083231834648599
50	$2 \cdot 3 \cdot 5^2 \cdot 13 \cdot 211 \cdot 20479$
	$p18$: 160189818494829241
	$p20$: 46218039785302111919
	$p44$: 19789860528346995527543912534464764790909391
51	3
	$p20$: 17708093685609923339
	$p73$
52	2^7
	$p17$: 43090793230759613
	$p76$
53	$3^3 \cdot 7^3$
	$p18$: 127534541853151177
	$p76$
54	$2 \cdot 3^6 \cdot 79 \cdot 389 \cdot 3167 \cdot 13309$
	$p11$: 69526661707
	$p22$: 8786705495566261913717
	$p51$
55	$5 \cdot 768643901$
	$p15$: 641559846437453
	$p22$: 1187847380143694126117
	$p55$
56	$2^2 \cdot 3$
	$p25$: 4324751743617631024407823 (BD)
	$p77$
57	$3 \cdot 17 \cdot 36769067$
	$p13$: 2205251248721
	$p37$: 2128126623795388466914401931224151279 (RB)
	$p47$: 14028351843196901173601082244449305344230057319
58	$2 \cdot 13$
	$p31$: 1448595612076564044790098185437 (BD)
	$p75$
59	3
	$p18$: 340038104073949513
	$p36$: 324621819487091567830636828971096713 (RB)
	$p55$
60	$2^3 \cdot 3 \cdot 5 \cdot 97 \cdot 157$
	$p103$
	continued...

n	Factors of Sm(n)
61	10386763
	$p14$: 35280457769357
	$p92$
62	$2 \cdot 3^2 \cdot 1709 \cdot 329167 \cdot 1830733$
	$p34$: 9703956232921821226401223348541281(TC)
	$p64$
63*	3^2
	$p11$: 17028095263
	$c105$
64	$2^2 \cdot 7 \cdot 17 \cdot 19 \cdot 197 \cdot 522673$
	$p19$: 1072389445090071307
	$p29$: 20203723083803464811983788589 (PW)
	$p60$
65*	$3 \cdot 5 \cdot 31 \cdot 83719$
	$c113$
66*	$2 \cdot 3 \cdot 7 \cdot 20143 \cdot 971077$
	$c111$
67	397
	$p18$: 183783139772372071
	$p105$
68*	$2^4 \cdot 3 \cdot 23 \cdot 764558869$
	$p10$: 1811890921
	$c105$
69	$3 \cdot 13 \cdot 23$
	$p22$: 8684576204660284317187
	$p24$: 281259608597535749175083
	$p32$: 15490495288652004091050327089107 (RB)
	$p49$: 3637485176043309178386946614318767365372143115591
70	$2 \cdot 5 \cdot 2411111$
	$p24$: 109315518091391293936799
	$p41$: 11555516101313335177332236222295571524323
	$p60$
71	$3^2 \cdot 83 \cdot 2281$
	$p31$: 7484379467407391660418419352839 (AM)
	$p95$
72	$2^2 \cdot 3^2 \cdot 5119$
	$p27$: 596176870295201674946617769 (BD)
	$p103$
73*	37907
	continued...

n	Factors of Sm(n)
	$c132$
74	$2 \cdot 3 \cdot 7 \cdot 1788313 \cdot 21565573$
	$p20$: 99014155049267797799
	$p25$: 1634187291640507800518363 (PW)
	$p31$: 1981231397449722872290863561307
	$p49$: 2377534541508613492655260491688014802698908815817
75*	$3 \cdot 5^2 \cdot 193283$
	$c133$
76	2^3
	$p18$: 828699354354766183
	$p27$: 213643895352490047310058981
	$p97$
77	3
	$p24$: 383481022289718079599637 (PW)
	$p24$: 874911832937988998935021
	$p39$: 164811751226239402858361187055939797929 (RB)
	$p58$
78*	$2 \cdot 3 \cdot 31 \cdot 185897$
	$c139$
79*	$73 \cdot 137$
	$p20$: 22683534613064519783
	$p24$: 132316335833889742191773
	$c102$
80	$2^2 \cdot 3^3 \cdot 5 \cdot 101 \cdot 10263751$
	$p25$: 1295331340195453366408489
	$p115$
81	$3^3 \cdot 509$
	$p30$: 152873624211113444108313548197 (AM)
	$p119$
82*	$2 \cdot 29 \cdot 4703 \cdot 10091$
	$p35$: 12295349967251726424104854676730107 (AM)
	$c111$
83*	$3 \cdot 53 \cdot 503$
	$p18$: 177918442980303859 (MF)
	$c134$
84	$2^5 \cdot 3$
	$p157$
85*	$5 \cdot 7^2$
	$c158$
	continued...

n	Factors of Sm(n)
86*	$2 \cdot 3 \cdot 23 \cdot 1056149$
	$c155$
87*	$3 \cdot 7 \cdot 231330259$
	$p10 : 4275444601$ (MF)
	$c145$
88*	2^2
	$p14 : 12414068351873$ (MF)
	$c153$
89*	$3 \cdot 3 \cdot 13 \cdot 31 \cdot 97 \cdot 163060459$
	$p18 : 789841356493369879$ (MF)
	$c137$
90*	$2 \cdot 3 \cdot 3 \cdot 5 \cdot 1987 \cdot 179827 \cdot 2166457$
	$c154$
91*	$37 \cdot 607$
	$p16 : 5713601747802353$ (MF)
	$p24 : 100397446615566314002487$ (MF)
	$c130$
92*	$2^3 \cdot 3 \cdot 75503$
	$c168$
93*	$3 \cdot 73 \cdot 1051$
	$p10 : 3298142203$ (MF)
	$c162$
94*	$2 \cdot 12871181$
	$p11 : 98250285823$ (MF)
	$c160$
95*	$3 \cdot 5 \cdot 7 \cdot 401$
	$c176$
96	$2 \cdot 2 \cdot 3 \cdot 23 \cdot 60331$
	$p175$
97	13
	$p183$
98*	$2 \cdot 3^2 \cdot 23 \cdot 37 \cdot 199$
	$p16 : 1495444452918817$ (MF)
	$c165$
99*	$3^2 \cdot 31601$
	$p12 : 786576340181$ (MF)
	$c171$
100*	$2^2 \cdot 5^2 \cdot 7^3 \cdot 8171 \cdot 1065829$
	$p10 : 2824782749$ (AM)

continued...

n	Factors of Sm(n)
	$p20$: 20317177407273276661 (MF)
	$c149$
101*	$3 \cdot 8377$
	$p21$: 799917088062980754649 (AM)
	$c169$
102	$2 \cdot 3 \cdot 19 \cdot 89 \cdot 3607 \cdot 15887 \cdot 32993$
	$p10$: 2865523753 (MF)
	$p172$
103*	$131 \cdot 1231$
	$p16$: 1713675826579469 (MF)
	$c180$
104*	$2^6 \cdot 3 \cdot 59 \cdot 773$
	$p20$: 19601852982312892289 (AM)
	$c177$
105*	$3 \cdot 5 \cdot 193$
	$p13$: 6942508281251 (MF)
	$c190$
106*	$2 \cdot 11 \cdot 127 \cdot 827$
	$c203$
107	3^3
	$p12$: 536288185369 (MF)
	$p199$
108*	$2^2 \cdot 3^3$
	$p18$: 128451681010379681 (AM)
	$c196$
109*	$7 \cdot 1559 \cdot 78176687$
	$p20$: 73024355266099724939 (AM)
	$c187$
110	$2 \cdot 3 \cdot 5 \cdot 4517$
	$p20$: 18443752916913621413 (AM)
	$p197$
111	$3 \cdot 293 \cdot 431 \cdot 230273 \cdot 209071 \cdot 241423723$
	$p10$: 3182306131 (MF)
	$p12$: 171974155987 (MF)
	$p13$: 1532064083461 (MF)
	$p17$: 59183601887848987 (MF)
	$p19$: 8526805649394145853 (AM)
	$p23$: 27151072184008709784271 (AM)
	$p109$
	continued...

n	Factors of $\mathrm{Sm}(n)$
112	$2^3 \cdot 16619 \cdot 449797 \cdot 894009023$
	$p17 : 74225338554790133$ (MF)
	$p23 : 10021106769497255963093$ (MF)
	$p169$
113*	$3 \cdot 11 \cdot 13 \cdot 5653 \cdot 1016453 \cdot 16784357$
	$p18 : 116507891014281007$ (AM)
	$p37 : 6844495453726387858061775603297883751$ (AM)
	$c157$
114*	$2 \cdot 3 \cdot 7 \cdot 178333$
	$c227$
115*	$5 \cdot 17 \cdot 19 \cdot 41 \cdot 36606 \cdot 71518987$
	$p18 : 283858194594979819$ (AM)
	$c202$
116*	$2^2 \cdot 3^2 \cdot 2239$
	$c235$
117*	$3^2 \cdot 31883$
	$p12 : 333699561211$ (MF)
	$p20 : 28437086452217952631$ (MF)
	$c206$
118*	$2 \cdot 83$
	$p11 : 33352084523$ (MF)
	$p20 : 20481677004050305811$ (MF)
	$c214$
119*	$3 \cdot 59 \cdot 101 \cdot 139 \cdot 2801$
	$c239$
120*	$2^4 \cdot 3 \cdot 5 \cdot 13 \cdot 16693063$
	$c241$
121*	278240783
	$c246$
122	$2 \cdot 3 \cdot 23 \cdot 618029123$
	$p14 : 31949422933783$ (MF)
	$p233$
123*	$3 \cdot 7 \cdot 37 \cdot 413923$
	$p10 : 1565875469$ (MF)
	$p16 : 5500432543504219$ (MF)
	$c227$
124*	$2^2 \cdot 739393$
	$p16 : 1958521545734977$ (MF)
	$c242$

continued...

n	Factors of Sm(n)
125*	$3^2 \cdot 5^3 \cdot 4019$
	$p13$: 7715697265127 (MF)
	$c247$
126	$2 \cdot 3^2 \cdot 29 \cdot 103 \cdot 70271$
	$p20$: 11513388742821485203 (MF)
	$p241$
127*	$53 \cdot 269 \cdot 4547$
	$p20$: 56560310643009044407 (AM)
	$c245$
128*	$2^3 \cdot 3 \cdot 7 \cdot 11 \cdot 59 \cdot 215329$
	$p22$: 8154316249498591416487 (MF)
	$c243$
129*	$3 \cdot 19$
	$c277$
130*	$2 \cdot 5$
	$p12$: 166817332889 (MF)
	$c269$
131*	$3 \cdot 19 \cdot 83 \cdot 1693$
	$p11$: 23210501651 (MF)
	$p12$: 575587270441 (MF)
	$c256$
132*	$2^2 \cdot 3 \cdot 79$
	$p13$: 2312656324607 (MF)
	$c272$
133*	$p19$: 8223519074965787731 (AM)
	$c272$
134*	$2 \cdot 3^3 \cdot 73 \cdot 6173$
	$p16$: 5527048386371021 (AM)
	$p28$: 1417349652747970442615118133 (AM)
	$c243$
135*	$3^3 \cdot 5 \cdot 11 \cdot 37 \cdot 647$
	$p10$: 1480867981 (MF)
	$p12$: 174496625453 (MF)
	$p15$: 151994480112757 (MF)
	$c255$
136*	$2^5 \cdot 1259 \cdot 4111$
	$p13$: 9485286634381 (MF)
	$p26$: 10151962417972135624157641 (AM)
	$c253$
	continued...

n	Factors of Sm(n)
137*	$3 \cdot 7^2$
	$p13$: 7459866979837 (MF)
	$c288$
138*	$2 \cdot 3 \cdot 181 \cdot 78311 \cdot 914569$
	$p15$: 413202386279227 (MF)
	$c277$
139*	13
	$p11$: 62814588973 (MF)
	$p12$: 115754581759 (MF)
	$p12$: 964458587927 (MF)
	$p22$: 9196988352200440482601 (MF)
	$c252$
140*	$2^2 \cdot 3 \cdot 5 \cdot 23 \cdot 761 \cdot 1873 \cdot 12841$
	$p11$: 34690415939 (MF)
	$p18$: 226556543956403897 (AM)
	$p23$: 10856300652094466205709 (AM)
	$c248$
141	$3 \cdot 107171$
	$p309$
142*	$2 \cdot 7 \cdot 4523 \cdot 14303 \cdot 76079$
	$p22$: 2244048237264532856611 (AM)
	$c282$
143*	$3^2 \cdot 859$
	$c317$
144	$2^3 \cdot 3^2 \cdot 6361$
	$p13$: 6585181700551 (MF)
	$p14$: 81557411089043 (MF)
	$p21$: 165684233831183308123 (MF)
	$p271$
145*	$5 \cdot 96151639$
	$c326$
146*	$2 \cdot 3 \cdot 13 \cdot 83$
	$p12$: 720716898227 (MF)
	$p19$: 1122016187632880261 (MF)
	$c296$
147*	$3 \cdot 59113$
	$p22$: 1833894252004152212837 (AM)
	$p31$: 1519080701040059055565669511153 (MF)
	$c276$
	continued...

n	Factors of Sm(n)
148*	$2^2 \cdot 197 \cdot 11927 \cdot 17377 \cdot 273131 \cdot 623321$
	$p13$: 3417425341307 (AM)
	$p13$: 4614988413949 (MF)
	$c288$
149*	$3 \cdot 103 \cdot 131 \cdot 1399$
	$c331$
150*	$2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 23$
	$p16$: 2315007810082921 (MF)
	$p26$: 92477662071402284092009799 (MF)
	$c296$
151*	$7 \cdot 53 \cdot 1801 \cdot 3323$
	$c335$
152*	$2^4 \cdot 3^2 \cdot 131 \cdot 10613$
	$p20$: 29354379044409991753 (AM)
	$p22$: 2587833772662908004979 (MF)
	$c298$
153*	$3^2 \cdot 29 \cdot 7237 \cdot 6987053 \cdot 8237263 \cdot 389365981$
	$c322$
154*	$2 \cdot 17 \cdot 19 \cdot 43$
	$p18$: 444802312089588077 (MF)
	$p21$: 855286987917657769927 (EW)
	$c311$
155	$3 \cdot 5 \cdot 66500999$
	$p24$: 223237752082537677918401 (EW)
	$p323$
156*	$2^2 \cdot 3 \cdot 7 \cdot 3307$
	$c354$
157*	$11 \cdot 53 \cdot 492601 \cdot 43169527$
	$p12$: 645865664923 (MF)
	$p18$: 125176035875938771 (MF)
	$c318$
158*	$2 \cdot 3 \cdot 17 \cdot 29 \cdot 53854663$
	$p21$: 164031369541076815133 (EW)
	$c334$
159*	$3 \cdot 71 \cdot 647$
	$p10$: 3175105177 (AM)
	$p25$: 1957802969152764074566129 (EW)
	$c330$
160*	$2^3 \cdot 5 \cdot 37 \cdot 130547 \cdot 859933 \cdot 21274133$

continued...

n	Factors of Sm(n)
	$p27$: 122800249349203273846720291 (EW)
	$c324$
161	$3^4 \cdot 59 \cdot 491 \cdot 81705851$
	$p360$
162*	$2 \cdot 3^5 \cdot 2999$
	$p21$: 393803780657062026421 (AM)
	$c351$
163*	2381
	$p11$: 72549525869 (AM)
	$p12$: 666733067809 (AM)
	$p25$: 1550529016982764630292633 (AM)
	$c330$
164*	$2^2 \cdot 3$
	$c383$
165*	$3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 247007767$
	$p15$: 490242053931613 (MF)
	$c359$
166	$2 \cdot 89$
	$p23$: 55566524959746113370037 (AM)
	$p365$
167*	$3 \cdot 3313$
	$c389$
168	$2^7 \cdot 3 \cdot 532709$
	$p387$
169*	$2671 \cdot 5233$
	$c392$
170*	$2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 701$
	$p14$: 73406007054077 (MF)
	$c382$
171*	$3^2 \cdot 1237$
	$p19$: 6017588157881558471 (AM)
	$c382$
172*	$2^2 \cdot 11 \cdot 13 \cdot 37$
	$c403$
173*	$3 \cdot 17 \cdot 53 \cdot 101 \cdot 153 \cdot 11633 \cdot 228673$
	$c394$
174*	$2 \cdot 3 \cdot 59 \cdot 277 \cdot 2522957$
	$p22$: 2928995151034569627547 (AM)
	$c381$

continued...

n	Factors of $\mathrm{Sm}(n)$
175*	5^2
	$p13 : 2606426254567$ (MF)
	$c403$
176*	$2^3 \cdot 3 \cdot 19 \cdot 1051$
	$p19 : 1031835687651103571$ (AM)
	$c396$
177*	$3 \cdot 109 \cdot 153277 \cdot 6690569$
	$p11 : 32545700623$ (MF)
	$p16 : 2984807754776597$ (MF)
	$c382$
178	2
	$p13 : 3144036216187$ (MF)
	$p17 : 11409535046513339$ (MF)
	$p397$
179*	$3^2 \cdot 7 \cdot 11 \cdot 359$
	$c423$
180*	$2^2 \cdot 3^2 \cdot 5 \cdot 43 \cdot 89 \cdot 7121$
	$c422$
181*	$31 \cdot 197 \cdot 70999$
	$p20 : 46096011552749697739$ (AM)
	$c406$
182*	$2 \cdot 3 \cdot 123529391$
	$c429$
183*	$3 \cdot 29 \cdot 661 \cdot 1723$
	$p16 : 3346484052265661$ (AM)
	$c417$
184*	$2^4 \cdot 7 \cdot 59 \cdot 191 \cdot 1093 \cdot 1223$
	$p11 : 22521973429$ (MF)
	$p17 : 15219125459582087$ (MF)
	$p18 : 158906425126963139$ (MF)
	$p19 : 2513521443592870099$ (MF)
	$c369$
185*	$3 \cdot 5 \cdot 94050577$
	$p13 : 4716042857821$ (MF)
	$p16 : 3479131875325867$ (MF)
	$c409$
186*	$2 \cdot 3 \cdot 1201$
	$p21 : 574850252802945786301$ (MF)
	$c425$
	continued...

n	Factors of $\mathrm{Sm}(n)$
187*	$349 \cdot 506442073$
	$c442$
188*	$2^2 \cdot 3^3$
	$c454$
189*	$3^3 \cdot 47 \cdot 1515169$
	$p10$: 1550882611 (MF)
	$p10$: 1687056803 (MF)
	$p21$: 348528133548561476953 (AM)
	$c410$
190	$2 \cdot 5 \cdot 379$
	$p23$: 46645758388308293907739 (AM)
	$p435$
191*	$3 \cdot 13 \cdot 5233$
	$p12$: 164130096629 (MF)
	$p20$: 13806214882775315521 (MF)
	$c429$
192*	$2^3 \cdot 3 \cdot 29 \cdot 41$
	$c463$
193*	$7 \cdot 419$
	$c467$
194*	$2 \cdot 3 \cdot 11 \cdot 31 \cdot 491 \cdot 34188439$
	$p14$: 28739332991401 (MF)
	$p16$: 8203347603076921 (MF)
	$p19$: 1507421050431503839 (MF)
	$p20$: 22805873052490568609 (MF)
	$p21$: 168560953170124281211 (MF)
	$c373$
195*	$3 \cdot 5 \cdot 397 \cdot 21728563 \cdot 300856949 \cdot 554551531$
	$p10$: 8174619091 (MF)
	$c438$
196	$2^2 \cdot 17 \cdot 73 \cdot 79$
	$p10$: 3834513037 (MF)
	$p465$
197*	$3^2 \cdot 37 \cdot 6277$
	$p16$: 1368971104990459 (MF)
	$c461$
198*	$2 \cdot 3^2 \cdot 7^2 \cdot 13$
	$c482$
199*	151
	continued...

n	Factors of Sm(n)
200*	$c487$ $2^5 \cdot 3 \cdot 5^2$ $c488$

Table 2: Factorizations of Sm(n), $1 < n \leq 200$

4.2 Reverse Smarandache Factors

Contributors of Reverse Smarandache factors		
RB	Robert Backstrom	bobb@atinet.com.au
BD	Bruce Dodson	bad0@lehigh.edu
MF	Micha Fleuren	michaf@sci.kun.nl
AM	Allan MacLeod	MACL-MS0@wpmail.paisley.ac.uk
RS	Ralf Stephan	stephan@tmt.destephan@tmt.de
PZ	Paul Zimmermann	Paul.Zimmermann@loria.fr
	continued...	

Table 3: Contributors of Reverse Smarandache factors

n	Factors of Rsm(n)
2	3.7
3	3.107
4	29.149
5	3.19.953
6	3.218107
7	19.402859
8	3^2.1997.4877
9	$3^2.17^2$.379721
10	7.28843.54421
11	3 $p12$: 370329218107
12	3.7 $p13$: 5767189888301
13	17.3243967.237927839
14	3.11.24769177 $p10$: 1728836281
15	$3.13.19^2$.79
	continued...

n	Factors Rsm(n)
	$p15$: 136133374970881
16	23.233.2531
	$p16$: 1190788477118549
17	3^2.13.17929.25411.47543.677181889
18	$3^2.11^2$.19.23.281.397.8577529.399048049
19	17.19
	$p13$: 1462095938449
	$p14$: 40617114482123
20	3.89.317.37889
	$p21$: 629639170774346584751
21	3.37
	$p12$: 732962679433
	$p19$: 2605975408790409767
22	13.137.178489
	$p13$: 1068857874509
	$p14$: 65372140114441
23	3.7.191
	$p32$: 578960862423763687712072079528211
24	3.107.457.57527
	$p28$: 28714434377387227047074286559
25	11.31.59.158820811.410201377
	$p20$: 19258319708850480997
26	3^3.929.1753.2503.4049.11171
	$p24$: 527360168663641090261567
27	3^5.83
	$p10$: 3216341629
	$p13$: 7350476679347
	$p18$: 571747168838911343
28	23.193.3061
	$p19$: 2150553615963932561
	$p21$: 967536566438740710859
29	3.11.709.105971.2901761
	$p10$: 1004030749
	$p24$: 405373772791370720522747
30	3.73.79.18041.24019.32749
	$p10$: 5882899163
	$p24$: 209731482181889469325577
31	7.30331061
	$p45$: 147434568678270777660714676905519165947320523
	continued...

n	Factors Rsm(n)
32	3.17.1231.28409
	$p12$: 103168496413
	$p35$: 17560884933793586444909640307424273
33	3.7.7349
	$p10$: 9087576403
	$p42$: 237602044832357211422193379947758321446883
34	89.488401.2480227.63292783.254189857
	$p10$: 3397595519
	$p19$: 5826028611726606163
35	3^2.881.1559.755173.7558043
	$p10$: 1341824123
	$p16$: 4898857788363449
	$p16$: 7620732563980787
36	$3^2.11^2$.971
	$p13$: 1114060688051
	$p22$: 1110675649582997517457
	$p24$: 277844768201513190628337
37	29.2549993
	$p20$: 39692035358805460481
	$p38$: 12729390074866695790994160335919964253
38	3.9833
	$p63$
39	3.19.73.709.66877
	$p58$
40	11.41.199
	$p27$: 537093776870934671843838337
	$p39$: 837983319570695890931247363677891299117
41	3.29.41.89.3506939
	$p14$: 18697991901857
	$p20$: 59610008384758528597
	$p28$: 3336615596121104783654504257
42	3.13249.14159.25073
	$p10$: 6372186599
	$p52$
43	52433
	$p20$: 73638227044684393717
	$p53$
44	3^2.7.3067.114883.245653
	$p23$: 65711907088437660760939
	continued...

n	Factors Rsm(n)
	$p41$: 12400566709419342558189822382901899879241
45	3^2.23.167.15859.25578743
	$p65$
46	23.35801
	$p12$: 543124946137
	$p23$: 45223810713458070167393
	$p43$: 2296875006922250004364885782761014060363847
47	3.11.31.59
	$p16$: 1102254985918193
	$p28$: 4808421217563961987019820401
	$p38$: 14837375734178761287247720129329493021
48	3.151.457.990013
	$p15$: 246201595862687
	$p24$: 636339569791857481119613
	$p39$: 150966139018567136078011449516167772467
49	71
	$p10$: 9777943361
	$p77$
50	3.157.3307
	$p13$: 3267926640703
	$p30$: 771765128032466758284258631297
	$p43$: 1285388803256371775298530192200584446319323
51	3.11
	$p92$
52	7.29.670001
	$p12$: 403520574901
	$p14$: 70216544961751
	$p16$: 1033003489172581
	$p47$: 13191839603253798296021585972083396625125257997
53	3^4.499.673.6287.57653.199236731
	$p16$: 1200017544380023
	$p28$: 1101541941540576883505692003
	$p31$: 2061265130010645250941617446327
54	$3^3.7^4$.13.1427.632778317
	$p11$: 57307460723
	$p13$: 7103977527461
	$p15$: 617151073326209
	$p43$: 2852320009960390860973654975784742937560247
55	357274517.460033621
	continued...

n	Factors Rsm(n)
	$p84$
56	3.13^2
	$p14$: 85221254605693
	$p87$
57	3.41
	$p11$: 25251380689
	$p93$
58	11.2425477
	$p15$: 178510299010259
	$p18$: 377938364291219561
	$p28$: 5465728965823437480371566249
	$p40$: 5953809889369952598561290100301076499293
59*	3
	$c109$
60	3
	$p10$: 8522287597
	$p101$
61	13.373
	$p22$: 6399032721246153065183
	$p42$: 214955646066967157613788969151925052620751 (RB)
	$p46$: 9236498149999681623847165427334133265556780913
62	3^2.11.487.6870011
	$p13$: 3921939670009
	$p14$: 11729917979119
	$p28$: 9383645385096969812494171823
	$p50$: 43792191037915584824808714186111429193335785529359
63	3^2.97.26347
	$p24$: 338856918508353449187667
	$p86$
64	397.653
	$p12$: 459162927787
	$p14$: 27937903937681
	$p24$: 386877715040952336040363
	$p65$
65*	3.7.23.13219.24371
	$c110$
66	3.53.83.2857.1154129.9123787
	$p103$
67*	43

continued...

n	Factors Rsm(n)
	$p11$: 38505359279
	$c113$
68	3.29.277213.68019179.152806439
	$p18$: 295650514394629363
	$p20$: 14246700953701310411
	$p67$
69	3.11.71.167.1481
	$p10$: 2326583863
	$p23$: 19962002424322006111361
	$p89$
70	1157237.41847137
	$p22$: 8904924382857569546497
	$p96$
71	3^2.17.131.16871
	$p10$: 1504047269
	$p11$: 82122861127
	$p19$: 1187275015543580261
	$p87$
72	3^2.449.1279
	$p129$
73	7.11.21352291
	$p10$: 1051174717
	$p17$: 92584510595404843
	$p20$: 33601392386546341921
	$p83$
74	3.177337
	$p10$: 6647068667
	$p11$: 31386093419
	$p15$: 669035576309897
	$p16$: 4313244765554839
	$p32$: 67415094145569534144512937880453 (PW)
	$p51$
75	3.7.230849.7341571.24260351
	$p10$: 1618133873
	$p14$: 19753258488427
	$p17$: 46752975870227777
	$p28$: 7784620088430169828319398031 (PW)
	$p53$
76*	53
	continued...

n	Factors Rsm(n)
	c142
77	3.919
	p15 : 571664356244249
	p22 : 6547011663195178496329 (PW)
	p27 : 591901089382359628031506373 (BD)
	p33 : 335808390273971395786635145251293 (PW)
	p46 : 3791725400705852972336277620397793613760330637
78*	3.17.47
	p14 : 17795025122047
	c131
79	160591
	p15 : 274591434968167
	p19 : 1050894390053076193
	p112
80*	3^3.11.443291.1575307
	p17 : 19851071220406859
	c121
81	3^3.23^2.62273.22193.352409
	p15 : 914359181934271 (MF)
	p120
82	*PRIME*! (RS)
83*	3
	c157
84*	3.11.47.83.447841.18360053
	p14 : 53294058577163 (MF)
	c130
85	p12 : 465619934881 (MF)
	p22 : 5013354844603778080337 (AM)
	p128
86*	3.7.3761.205111.16080557.16505767
	c139
87	3.2423
	p25 : 4433139632126658657934801 (AM)
	p30 : 951802198132419645688492825211 (MF)
	p107
88*	73.8747
	c162
89*	3^2.19.7052207
	c161
	continued...

n	Factors Rsm(n)
90*	3^2.157.257.691
	c140
91*	11.29.163.3559.2297.22899893
	p15 : 350542343218231 (MF)
	p25 : 8365221234379371317434883 (MF)
	c115
92*	3.17.113.376589.3269443.6872137
	c153
93*	3.13.69317.14992267
	c164
94*	7.593.18307
	p11 : 51079607083 (MF)
	c161
95*	3.11.13.53.157.623541439
	c166
96*	3.7.211.14563.2297
	c172
97*	1553
	c182
98	3^2.101.401.5741.375373
	p173
99*	3^2.109.41829209
	p12 : 174489586693 (MF)
	c168
100*	13.6779
	p11 : 48856332919 (MF)
	p26 : 41858129936073024200781901 (MF)
	c150
101*	3
	p11 : 16320902651 (MF)
	p19 : 3845388775716560041 (MF)
	p33 : 693173763848292948494434792706137 (AM)
	c132
102*	3.101.103.36749
	p11 : 10189033219 (MF)
	p20 : 23663501701518727831 (AM)
	p26 : 52648894306108287380398039 (AM)
	c133
103*	19.29.103.3119.154009291
	continued...

n	Factors Rsm(n)
90*	3^2.157.257.691
	c140
91*	11.29.163.3559.2297.22899893
	p15 : 350542343218231 (MF)
	p25 : 8365221234379371317434883 (MF)
	c115
92*	3.17.113.376589.3269443.6872137
	c153
93*	3.13.69317.14992267
	c164
94*	7.593.18307
	p11 : 51079607083 (MF)
	c161
95*	3.11.13.53.157.623541439
	c166
96*	3.7.211.14563.2297
	c172
97*	1553
	c182
98	3^2.101.401.5741.375373
	p173
99*	3^2.109.41829209
	p12 : 174489586693 (MF)
	c168
100*	13.6779
	p11 : 48856332919 (MF)
	p26 : 41858129936073024200781901 (MF)
	c150
101*	3
	p11 : 16320902651 (MF)
	p19 : 3845388775716560041 (MF)
	p33 : 693173763848292948494434792706137 (AM)
	c132
102*	3.101.103.36749
	p11 : 10189033219 (MF)
	p20 : 23663501701518727831 (AM)
	p26 : 52648894306108287380398039 (AM)
	c133
103*	19.29.103.3119.154009291
	continued...

n	Factors Rsm(n)
	$p12$: 329279243129 (MF)
	$p13$: 1240336674347 (MF)
	$c161$
104*	3.7.60953.1890719
	$p11$: 10446899741 (MF)
	$p15$: 216816630080837 (MF)
	$p19$: 1614245774588631629 (MF)
	$c149$
105*	3.7.859.6047.63601
	$c194$
106*	$p22$: 1912037972972539041647 (AM)
	$p22$: 3052818746214722908609 (AM)
	$c167$
107*	3^3.13.4519.114967
	$p10$: 1425213859 (MF)
	$p14$: 17641437858251 (MF)
	$c179$
108	3^3.23.457.1373
	$p12$: 605434593221 (MF)
	$p12$: 703136513561 (MF)
	$p183$
109	11.29.31^2.1709.30345569
	$p14$: 42304411918757 (MF)
	$p189$
110*	3.11.19.53.229.24672421
	$p24$: 611592384837948878235019 (AM)
	$c183$
111*	3.61.269.470077.143063.544035253
	$c200$
112*	137
	$p12$: 262756224547 (MF)
	$c214$
113*	3.19.45061.111211
	$c219$
114*	3.19.53.59
	$c228$
115*	137.509.1720003
	$c226$
116	3^2.83.103.156307.176089.21769127
	continued...

n	Factors Rsm(n)
	$p217$
117	3^2
	$p242$
118	7.4603
	$p241$
119*	3.7
	$c247$
120*	3.73
	$c249$
121*	31.371177
	$c248$
122*	3.17
	$p11$: 91673873887 (MF)
	$c245$
123	3.1197997
	$p11$: 15744706711 (MF)
	$p244$
124*	37.1223
	$c259$
125	3^2.59.83
	$p10$: 5961006911 (MF)
	$p13$: 1096598255677 (MF)
	$p240$
126*	3^2.13.68879.135342173
	$c255$
127*	97
	$p16$: 1385409249340483 (AM)
	$c255$
128*	3.34613.29497667
	$c263$
129*	3.23.1213.82507
	$p12$: 420130412231 (MF)
	$c257$
130*	31.263.86969.642520369
	$c264$
131*	3.11.4111.852143
	$p12$: 606617222863 (MF)
	$p23$: 33247682213571703426139 (AM)
	$c239$
	continued...

n	Factors Rsm(n)
132	3.7.11.41.43.31259.69317.180307.199313
	$p17$: 16995472858509251 (MF)
	$p20$: 56602777258539682957 (AM)
	$p226$
133	7.13
	$p20$: 22533511116338912411 (AM)
	$p269$
134*	3^3.37.29004967
	$p17$: 60164048964096599 (AM)
	$c266$
135*	3^3.211.5393.98563
	$p12$: 207481965329 (MF)
	$p22$: 6789282931372049267693 (AM)
	$c251$
136*	
137*	3.179
	$p22$: 6796599525965619205571 (AM)
	$c278$
138*	3.119611.314087617
	$c292$
139*	
140*	3.317.772477
	$p15$: 153629260660723 (AM)
	$c289$
141*	3.631.65831
	$c307$
142*	859.2377.2909.6521
	$p14$: 41190901651547 (MF)
	$c291$
143	3^2.93971
	$p12$: 9053448211979 (MF)
	$p302$
144*	3^2
	$p19$: 5028055908018884749 (MF)
	$c304$
145*	57719.2691841
	$p20$: 45690580335973653419 (MF)
	$c296$
146*	3.7^2.277.19319.55807
	continued...

n	Factors Rsm(n)
	$p13$: 2454423915989 (MF)
	$c304$
147*	$3.7^2.19.31.15467623$
	$c321$
148*	$p20$: 33825333713396366003 (AM)
	$p23$: 25082957895838310384953 (AM)
	$c294$
149*	3.109.34442413
	$c329$
150*	3.59.257
	$c337$
151*	$p10$: 7134941903 (MF)
	$c335$
152	$3^2.13$
	$p21$: 412891312089439668533 (MF)
	$p325$
153*	$3^2.67793$
	$p18$: 237333508084627139 (MF)
	$c328$
154*	11.53861
	$p10$: 1118399729 (MF)
	$c339$
155*	3.41.33842293
	$c347$
156*	3.21961
	$c355$
157*	$p10$: 4136915059 (MF)
	$c353$
158*	3.31.89209
	$p10$: 1379633699 (MF)
	$p14$: 54957888020501 (MF)
	$c336$
159*	3.13.5669.11213.816229087
	$p10$: 50611041883 (MF)
	$c340$
160*	7.942037.1223207
	$p21$: 125729584994875519171 (AM)
	$c339$
161*	$3^7.7.37.67.6521.826811.6018499$
	continued...

n	Factors Rsm(n)
	$p23$: 7755890044266075256801 (MF)
	$c328$
162*	3^4.1295113.202557967
	$c361$
163*	$p16$: 1139924663537993 (MF)
	$p17$: 17672171439068059 (MF)
	$c350$
164	3.193
	$p24$: 105444241520715055381519 (AM)
	$p358$
165*	3
	$c386$
166*	$p15$: 396444477663149 (MF)
	$p32$: 15221332593310506150048824812249 (AM)
	$c344$
167*	3.17.373.7346281.8927551.194571659
	$p20$: 68277637362521294401 (AM)
	$c347$
168*	3.59.35537.68102449
	$p19$: 7766035514845504007 (AM)
	$c362$
169*	
170	3^2.23.
	$p16$: 3737994294192383 (MF)
	$p384$
171*	3^2.37
	$p12$: 237089136881 (MF)
	$p19$: 2153684224509566597 (MF)
	$p21$: 175530075465216996787 (MF)
	$p22$: 8105319358780665120301 (MF)
	$c330$
172	17.29.281
	$p10$: 4631571401 (MF)
	$p11$: 31981073881 (MF)
	$p15$: 119749047957053 (MF)
	$p368$
173*	3.1787
	$c407$
174*	3.7.269.397.156894809
	continued...

n	Factors Rsm(n)
	$c399$
175*	7.11
	$p10$: 3763462823 (MF)
	$c405$
176*	3.11.47.49613
	$p13$: 2800890701267 (MF)
	$p15$: 315698062297249 (MF)
	$p27$: 880613122533775176075766757 (MF)
	$c358$
177	3.73.1753
	$p14$: 29988562180903 (MF)
	$p404$
178	13.47.353.644951.487703.1436731
	$p12$: 728961984851 (MF)
	$p14$: 34686545199997 (MF)
	$p14$: 36329334000803 (MF)
	$p364$
179*	3^2.23.43
	$p14$: 50981967790529 (MF)
	$c411$
180*	3^2.29
	$p17$: 33644294710009721 (MF)
	$c413$
181*	325251083
	$p17$: 57421731284347247 (MF)
	$c410$
182*	3.107.5568133
	$p12$: 139065644033 (MF)
	$c417$
183*	3.23.89
	$c437$
184*	23.19531
	$p15$: 196140464783429 (MF)
	$c424$
185*	3.13.919
	$p11$: 32173266383 (MF)
	$c432$
186*	3.23
	$c448$
	continued...

n	Factors Rsm(n)
187*	61.83.103.523.3187
	$p19$: 1018598504636281577 (MF)
	$c423$
188*	3^3.7.7681.65141
	$c445$
189*	3^3.7.2039.3823.9739.212453.10586519
	$c433$
190*	83.107.1871.25346653
	$c447$
191	3.809
	$p18$: 627089953107590081 (MF)
	$p444$
192*	3.2549
	$c464$
193*	47.503.12049
	$c463$
194*	3.179
	$p22$: 8000103240831609636731 (AM)
	$p23$: 77947886830169946060329 (MF)
	$c426$
195*	3.79.8219
	$c471$
196*	19
	$p16$: 8982588119304797 (AM)
	$c463$
197*	3^2.11.43.11743.125201.867619
	$p11$: 61951529111 (MF)
	$p14$: 27090970290157 (MF)
	$c440$
198	3^2.11.37.2837
	$p19$: 1245013373736039779 (MF)
	$p461$
199*	103.2377
	$c484$
200*	3.1666421
	$c485$

Table 4: Factorizations of Rsm(n), $1 < n \leq 200$

5 Acknowledgements

I'd like to thank Ralf Stephan for his previous work on these numbers. Also I would like to thank Allan MacLeod in particular, for he has contributed

a load of factors to these lists, as well as some primality proofs. All other contributers earn a warm thank you here too.

Thanks must also go to Paul Zimmermann, Torbjörn Granlund, François Morain and Yuji Kida for their wonderfull free programs.

Any additions are very much welcomed and can be send to the following email-adress: `michaf@sci.kun.nl`.

The author can also be reached at the following address: Micha Fleuren, Acaciastraat 16, 6598 BC, Heijen, The Netherlands.

The very latest up to date representation of this list can be found at the next URL: `http://www.sci.kun.nl/sigma/Persoonlijk/michaf/ecm/`.

References

[Ad] L.M. Adleman, C. Pomerance and R.S. Rumely: *On Distinguishing Prime Numbers from Composite Number*, Ann. Math. **117**, 173-206 (1983)

[At] A.O. Atkin and F. Morain: *Elliptic curves and primality proving*, Math. Comp. **60**, 399-405 (1993)

[Co] H. Cohen and A.K. Lenstra: *Primality Testing and Jacobi Sums*, Math. Comp. **42**, 297-330 (1984)

[Gr] T. Granlund: *GNU Multiple Precision Arithmetic Library*, `http://www.matematik.su.se/~tege/gmp/`

[Ki] Y.Kida: *UBASIC program*, `ftp://rkmath.rikkyo.ac.jp/ubibm/`

[Le] H.W. Lenstra, Jr: *Factoring integers with elliptic curves*, Annals of Mathematics (2) **126**, 649-673 (1987)

[Mo] F. Morain: *Elliptic Curve Primality Proving*, `http://pauillac.inria.fr/algo/morain/`

[Si] R.D. Silverman: *The Multiple Polynomial Quadratic Sieve*, Math. Comp. **48**, 329-339 (1987)

[Sm] F. Smarandache: *Only Problems, Not Solutions!*, Xiquan Publ., Phoenix-Chicago, (1993)

[St] R.W. Stephan: *Factors and primes in two Smarandache sequences*, Smarandache Notion Journal 9, 4-10 (1998)

[Zi] P. Zimmermann: *The ECM-NET Project*,
`http://www.loria.fr/~zimmerma/records/ecmnet.html`

Smarandache Continued Fractions

Henry Ibstedt[1]

Abstract: The theory of general continued fractions is developed to the extent required in order to calculate Smarandache continued fractions to a given number of decimal places. Proof is given for the fact that Smarandache general continued fractions built with positive integer Smarandache sequences having only a finite number of terms equal to 1 is convergent. A few numerical results are given.

Introduction

The definitions of Smarandache continued fractions were given by Jose Castillo in the Smarandache Notions Journal, Vol. 9, No 1-2 [1].

A Smarandache Simple Continued Fraction is a fraction of the form:

$$a(1)+\cfrac{1}{a(2)+\cfrac{1}{a(3)+\cfrac{1}{a(4)+\cfrac{1}{a(5)+\cdots}}}}$$

where a(n), for n≥1, is a Smarandache type Sequence, Sub-Sequence or Function.

Particular attention is given to the Smarandache General Continued Fraction defined as

$$a(1)+\cfrac{b(1)}{a(2)+\cfrac{b(2)}{a(3)+\cfrac{b(3)}{a(4)+\cfrac{b(4)}{a(5)+\cdots}}}}$$

where a(n) and b(n), for n≥1, are Smarandache type Sequences, Sub-Sequences or Functions.

[1] HIbstedt@swipnet.se

As a particular case the following example is quoted

$$1+\cfrac{1}{12+\cfrac{21}{123+\cfrac{321}{1234+\cfrac{4321}{12345+\cdots}}}}$$

Here 1, 12, 123, 1234, 12345, ... is the Smarandache Consecutive Sequences and 1, 21, 321, 4321, 54321, ... is the Smarandache Reverse Sequence.

The interest in Castillo's article is focused on the calculation of such fractions and their possible convergens when the number of terms approaches infinity. The theory of simple continued fractions is well known and given in most standard textbooks in Number Theory. A very comprehensive theory of continued fractions, including general continued fractions is found in *Die Lehre von den Kettenbrüchen* [2]. The symbols used to express facts about continued fractions vary a great deal. The symbols which will be used in this article correspond to those used in Hardy and Wright *An Introduction to the Theory of Numbers* [3]. However, only simple continued fractions are treated in the text of Hardy and Wright. Following more or less the same lines the theory of general continued fractions will be developed in the next section as far as needed to provide the necessary tools for calculating Smarandache general continued fractions.

General Continued Fractions

We define a finite general continued fraction through

$$C_n = q_0+\cfrac{r_1}{q_1+\cfrac{r_2}{q_2+\cfrac{r_3}{q_3+\cfrac{r_4}{q_4+\ldots}}}} = q_0+\frac{r_1}{q_1+}\,\frac{r_2}{q_2+}\,\frac{r_3}{q_3+}\,\frac{r_3}{q_4+}\,\cdots\,\frac{r_n}{q_n} \tag{1}$$

where $\{q_0, q_1, q_2, \ldots. q_n\}$ and $\{r_1, r_2, r_3 \ldots r_n\}$ are integers which we will assume to be positive.

The above definition is an extension of the definition of a simple continued fraction where $r_1=r_2= \ldots =r_n=1$. The theory developed here will apply to simple continued fractions as well by replacing r_k ($k=1, 2, \ldots$) in formulas by 1 and simply ignoring the reference to r_k when not relevant.

The formula (1) will usually be expressed in the form

$$C_n=[q_0,q_1,q_2,q_3, \ldots q_n,r_1,r_2,r_3 \ldots r_n] \tag{2}$$

For a simple continued fraction we would write

$$C_n=[q_0,q_1,q_2,q_3, \dots q_n] \tag{2'}$$

If we break off the calculation for m≤n we will write

$$C_m=[q_0,q_1,q_2,q_3, \dots q_m,r_1,r_2,r_3 \dots r_m] \tag{3}$$

Equation (3) defines a sequence of finite general continued fractions for m=1, m=2, m=3, … . Each member of this sequence is called a **convergent** to the continued fraction

Working out the general continued fraction in stages, we shall obviously obtain expressions for its convergents as quotients of two sums, each sum comprising various products formed with q_0, q_1, q_2, $\dots q_m$ and r_1, r_2, $\dots r_m$.

If m=1, we obtain the first convergent

$$C_1=[q_0,q_1,r_1]=q_0+\frac{r_1}{q_1}=\frac{q_0q_1+r_1}{q_1} \tag{4}$$

For m=2 we have

$$C_2=[q_0,q_1,q_2,r_1,r_2]=q_0+\frac{q_2r_1}{q_1q_2+r_2}=\frac{q_0q_1q_2+q_0r_2+q_2r_1}{q_1q_2+r_2} \tag{5}$$

In the intermediate step the value of $q_1+\frac{r_2}{q_2}$ from the previous calculation has been quoted, putting q_1, q_2 and r_2 in place of q_0, q_1 and r_1. We can express this by

$$C_2=[q_0,[q_1,q_2,r_2],r_1] \tag{6}$$

Proceeding in the same way we obtain for m=3

$$C_3=[q_0,q_1,q_2,q_3,r_1,r_2,r_3]=q_0+\frac{(q_2q_3+r_3)r_1}{q_1q_2q_3+q_1r_3+q_3r_2}=$$

$$\frac{q_0q_1q_2q_3+q_0q_1r_3+q_0q_3r_2+q_2q_3r_1+r_1r_3}{q_1q_2q_3+q_1r_3+q_3r_2} \tag{7}$$

or generally

$$C_m=[q_0,q_1,\dots q_{m-2},[q_{m-1},q_m,r_m],r_1,r_2,\dots r_{m-1}] \tag{8}$$

which we can extend to

$$C_n=[q_0,q_1,\dots q_{m-2},[q_{m-1},q_m,\dots q_n,r_m,\dots r_n],r_2,r_2,\dots r_{m-1}] \tag{9}$$

Theorem 1:

Let A_m and B_m be defined through

$$A_0=q_0, \quad A_1=q_0q_1+r_1, \qquad A_m=q_mA_{m-1}+r_mA_{m-2} \quad (2\le m\le n)$$

$$B_0=1, \quad B_1=q_1, \qquad B_m=q_mB_{m-1}+r_mB_{m-2} \quad (2\le m\le n) \tag{10}$$

then $C_m=[q_0,q_1,\ldots q_m,r_1,\ldots r_m] = \dfrac{A_m}{B_m}$, i.e. $\dfrac{A_m}{B_m}$ is the *mth* convergent to the general continued fraction.

Proof: The theorem is true for m=0 and m=1as is seen from $[q_0]=\dfrac{q_0}{1}=\dfrac{A_0}{B_0}$ and $[q_0,q_1,r_1]=\dfrac{q_0q_1+r_1}{q_1}=\dfrac{A_1}{B_1}$. Let us suppose that it is true for a given m<n. We will induce that it is true for m+1

$$\begin{aligned}
[q_0,q_1,\ldots q_{m+1},r_1,\ldots r_{m+1}] &= [q_0,q_1,\ldots q_{m-1},[q_m,q_{m+1},r_{m+1}],r_1,\ldots r_m] \\
&= \frac{[q_m,q_{m+1},r_{m+1}]A_{m-1}+r_mA_{m-2}}{[q_m,q_{m+1},r_{m+1}]B_{m-1}+r_mB_{m-2}} \\
&= \frac{(q_m+\frac{r_{m+1}}{q_{m+1}})A_{m-1}+r_mA_{m-2}}{(q_m+\frac{r_{m+1}}{q_{m+1}})B_{m-1}+r_mB_{m-2}} \\
&= \frac{q_{m+1}(q_mA_{m-1}+r_mA_{m-2})+r_{m+1}A_{m-1}}{q_{m+1}(q_mB_{m-1}+r_mB_{m-2})+r_{m+1}B_{m-1}} \\
&= \frac{q_{m+1}A_{m-1}+r_{m+1}A_{m-1}}{q_{m+1}B_{m-1}+r_{m+1}B_{m-1}}=\frac{A_{m+1}}{B_{m+1}}
\end{aligned}$$

□

The recurrence relations (10) provide the basis for an effective computer algorithm for successive calculation of the convergents C_m.

Theorem 2:

$$A_mB_{m-1}-B_mA_{m-1}=(-1)^{m-1}\prod_{k=1}^{m} r_k \tag{11}$$

Proof: For m=1 we have $A_1B_0-B_1A_0=q_0q_1+r_1-q_0q_1=r_1$.

$$A_mB_{m-1}-B_mA_{m-1}=(q_mA_{m-1}+r_mA_{m-2})B_{m-1}-(q_mB_{m-1}+r_mB_{m-2})A_{m-1}=$$
$$-r_m(A_{m-1}B_{m-2}-B_{m-1}A_{m-2})$$

By repeating this calculation with m-1, m-2, ..., 2 in place of m, we arrive at

$$A_mB_{m-1}-B_mA_{m-1}= \ldots = (A_1B_0-B_1A_0)(-1)^{m-1}\prod_{k=2}^{m} r_k = (-1)^{m-1}\prod_{k=1}^{m} r_k$$

□

Theorem 3:

$$A_mB_{m-2}-B_mA_{m-2}=(-1)^m q_m \prod_{k=1}^{m-1} r_k \tag{12}$$

Proof: This theorem follows from theorem 3 by inserting expressions for A_m and B_m

$$A_mB_{m-2}-B_mA_{m-2}=(q_mA_{m-1}+r_mA_{m-2})B_{m-2}-(q_mB_{m-1}+r_mB_{m-2})A_{m-2}=$$
$$q_m(A_{m-1}B_{m-2}-B_{m-1}A_{m-2})=(-1)^m q_m \prod_{k=1}^{m-1} r_k$$

□

Using the symbol $C_m=\frac{A_m}{B_m}$ we can now express important properties of the number sequence C_m, m=1, 2, ... , n. In particular we will be interested in what happens to C_n as n approaches infinity.

From (11) we have

$$C_n - C_{n-1} = \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} = \frac{(-1)^{n-1}\prod_{k=1}^{n} r_k}{B_{n-1}B_n} \tag{13}$$

while (12) gives

$$C_n - C_{n-2} = \frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}} = \frac{(-1)^{n-1} q_n \prod_{k=1}^{n-1} r_k}{B_{n-2}B_n} \tag{14}$$

We will now consider infinite positive integer sequences $\{q_0, q_1, q_2, \ldots\}$ and $\{r_1, r_2, \ldots\}$ where only a finite number of terms are equal to 1. This is generally the case for Smarandache sequences. We will therefore prove the following important theorem.

Theorem 4:

A general continued fraction for which the sequences $q_0, q_1, q_2, \ldots$ and $r_1, r_2, \ldots$ are positive integer sequences with at most a finite number of terms equal to 1 is convergent.

Proof: We will first show that the product $B_{n-1}B_n$, which is a sum of terms formed by various products of elements from $\{q_1, q_2, \ldots q_n, r_1, r_2, \ldots r_{n-1}\}$, has one term which is a multiple of $\sum_{k=2}^{n} r_k$. Looking at the process by which we calculated C_1, C_2, and C_3, equations

4, 5 and 7, we see how terms with the largest number of factors r_k evolve in numerators and denominators of C_k. This is made explicit in figure 1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
Num. A_m	r_1	q_0r_2	r_1r_3	$q_0r_2r_4$	$r_1r_3r_5$	$q_0r_2r_4r_6$	$r_1r_3r_5r_7$	$q_0r_2r_4r_6r_8$
Den. B_m	-	r_2	q_1r_3	r_2r_4	$q_1r_3r_5$	$r_2r_4r_6$	$q_1r_3r_5r_7$	$r_2r_4r_6r_8$

Figure 1. The terms with the largest number of r-factors in numerators and denominators.

As is seen from figure 1 two consecutive denominators B_nB_{n-1} will have a term with $r_2r_3 \ldots r_n$ as factor. This means that the numerator of (13) will not cause C_n-C_{n-1} to diverge. On the other hand $B_{n-1}B_n$ contains the term $(q_1q_2 \ldots q_{n-1})^2q_n$ which approaches ∞ as $n\to\infty$. It follows that $\lim_{n\to\infty}(C_n - C_{n-1}) = 0$.

From (14) we see that

1. If n is odd, say n=2k+1, than $C_{2k+1} < C_{2k-1}$ forming a monotonously decreasing number sequence which is bounded below (positive terms). It therefore has limit.
   $\lim_{k\to\infty} C_{2k+1} = C_1$.
2. If n is even, n=2k, than $C_{2k} > C_{2k-2}$ forming a monotonously increasing number sequence. This sequence has an upper bound because $C_{2k} < C_{2k+1} \to C_1$ $as\, k\to\infty$. It therefore has limit.
   $\lim_{k\to\infty} C_{2k} = C_2$.
3. Since $\lim_{n\to\infty}(C_n - C_{n-1}) = 0$ we conclude that $C_1=C_2$. Consequently $\lim_{n\to\infty} C_n = C$ exists.

□

Calculations

A *UBASIC* program has been developed to implement the theory of Smarandache general continued fractions. The same program can be used for classical continued fractions since these correspond to the special case of a general continued fraction where $r_1=r_2= \ldots =r_n=1$.

The complete program used in the calculations is given below. The program applies equally well to simple continued fractions by setting all element of the array R equals to 1.

```
10 point 10
20 dim Q(25),R(25),A25),B25)
30 input "Number of decimal places of accuracy: ";D
40 input "Number of input terms for R (one more for Q) ";N%
50 cls
60 for I%=0 to N%:read Q(I%):next
70 data                                    'The relevant data q0, q1, ...
80 for I%=1 to N%:read R(I%):next
90 data                                    'The relevant data for r1,r2, ...
100 print tab(10);"Smarandache Generalized Continued Fraction"
110 print tab(10);"Sequence Q:";
120 for I%=0 to 6:print Q(I%);:next:print " ETC"
130 print tab(10);"Sequence R:";
140 for I%=1 to 6:print R(I%);:next:print " ETC"
```

```
150 print tab(10);"Number of decimal places of accuracy: ";D
160 A(0)=Q(0):B(0)=1                                    'Initiating recurrence algorithm
170 A(1)=Q(0)*Q(1)+R(1):B(1)=Q(1)
180 Delta=1:M=1                                         'M=loop counter
190 while abs(Delta)>10^(-D)                            'Convergens check
200 inc M
210 A(M)=Q(M)*A(M-1)+R(M)*A(M-2)                        'Recurrence
220 B(M)=Q(M)*B(M-1)+R(M)*B(M-2)
230 Delta=A(M)/B(M)-A(M-1)/B(M-1)                       'Cm-Cm-1
240 wend
250 print tab(10);"An/Bn=";:print using(2,20),A(M)/B(M) 'Cn in decimalform
260 print tab(10);"An/Bn=";:print A(M);"/";B(M)         'Cn in fractional form
270 print tab(10);"Delta=";:print using(2,20),Delta;    'Delta=Last difference
280 print " for n=";M                                   'n=number of iterations
290 print
300 end
```

To illustrate the behaviour of the convergents C_n have been calculated for $q_1=q_2= \ldots = q_n=1$ and $r_1=r_2= \ldots = r_n=10$. The iteration of C_n is stopped when $\Delta_n= |C_n-C_{n-1}|<0.01$. Table 1 shows the result which is illustrated in figure 2. The factor $(-1)^{n-1}$ in (13) produces an oscillating behaviour with diminishing amplitude approaching $\lim_{n\to\infty} C_n = C$

Table 1. Value of convergents C_n for $q\varepsilon\{1,1,\ldots\}$ and $r\varepsilon\{10,10,\ldots\}$

n	1	2	3	4	5	6	7	8	9	10	11
C_n	11	1.91	6.24	2.6	4.84	3.07	4.26	3.35	3.99	3.51	3.85
n	12	13	14	15	16	17	18	19	20	21	22
C_n	3.6	3.78	3.65	3.74	3.67	3.72	3.69	3.71	3.69	3.71	3.7

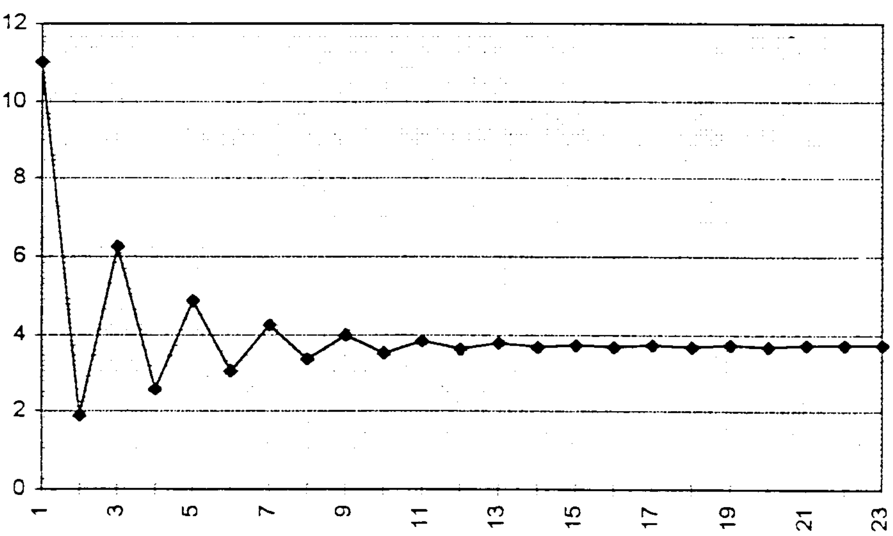


Figure 2. C_n as a function of n

A number of sequences, given below, will be substituted into the recurrence relations (10) and the convergence estimate (13).

$S_1=\{1, 1, 1, \ldots\ldots\}$
$S_2=\{1, 2, 1, 2, 1, 2, \ldots\ldots\}$
$S_3=\{3, 3, 3, 3, 3, 3, \ldots\ldots\}$
$S_4=\{1, 12, 123, 1234, 12345, 123456, \ldots..\}$ Smarandache Consecutive Sequence.
$S_5=\{1, 21, 321, 4321, 54321, 654321, \ldots..\}$ Smarandache Reverse Sequence.
CS1={1,1,2,8,9,10,512,513,514,520,521,522,729,730,731,737,738, ...
NCS1={1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,29,30,...

The Smarandache CS1 sequence definition: CS1(n) is the smallest number, strictly greater than the previous one (for $n\geq3$), which is the cubes sum of one or more previous distinct terms of the sequence.
The Smarandache NCS1 sequence definition: NCS1(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence.
These sequences have been randomly chosen form a large number of Smarandache sequences [5].

As expected the last fraction in table 2 converges much slower than the previous one. These general continued fractions are, of course, very artificial as are the sequences on which they are based. As is often the case in empirical number theory it is not the individual figures or numbers which are of interest but the general behaviour of numbers and sequences under certain operations. In the next section we will carry out some experiments with simple continued fractions.

Experiments with Simple Continued Fractions

The theory of simple continued fractions is covered in standard textbooks. Without proof we will therefore make use of some of this theory to make some more calculations. We will first make use of the fact that

> *There is a one to one correspondence between irrational numbers and infinite simple continued fractions.*

The approximations given in table 2 expressed as simple continued fractions would therefore show how these are related to the corresponding general continued fractions.

Table 2. Calculation of general continued fractions

Q	R	n	Δ_n	C_n (dec.form)	C_n (fraction)
S_1	S_1	18	$-9\cdot10^{-8}$	1.6180339	$\frac{6765}{4181}$
S_2	S_1	13	$8\cdot10^{-8}$	1.3660254	$\frac{7953}{5822}$
S_2	S_3	22	$-9\cdot10^{-8}$	1.8228756	$\frac{1402652240}{769472267}$
S_4	S_1	2	$-7\cdot10^{-6}$	1.04761	$\frac{7063}{6742}$
		3	$5\cdot10^{-12}$	1.04761198457	$\frac{30519245}{29132203}$
		4	$-2\cdot10^{-20}$	1.0476119845794017019	$\frac{1657835914708}{1582490405905}$
S_4	S_5	2	$-1\cdot10^{-3}$	1.082	$\frac{540}{499}$
		4	$-7\cdot10^{-10}$	1.082166760	$\frac{8245719435}{7619638429}$
		6	$-1\cdot10^{-19}$	1.0821667605141670276 8	$\frac{418939686644589150004}{387130433063328840289}$
S_5	S_1	2	$-7\cdot10^{-6}$	1.04761	$\frac{7063}{6742}$
		3	$5\cdot10^{-12}$	1.04761198457	$\frac{30519245}{29132203}$
		4	$-2\cdot10^{-20}$	1.04761198457940170194	$\frac{1657835914708}{1582490405905}$
S_5	S_4	2	$-8\cdot10^{-5}$	1.0475	$\frac{2358}{2251}$
		3	$7\cdot10^{-9}$	1.04753443	$\frac{2547455}{2431858}$
		5	$1\cdot10^{-20}$	1.04753443663236268392	$\frac{60363763803209222}{57624610411155561}$
CS1	NCS1	6	$-1\cdot10^{-7}$	1.540889	$\frac{1376250}{893153}$
		7	$3\cdot10^{-12}$	1.54088941088	$\frac{1412070090}{916399373}$
		9	$-1\cdot10^{-20}$	1.54088941088788795255	$\frac{377447939426190}{244954593599743}$
NCS1	CS1	16	$-5\cdot10^{-5}$	0.6419	$\frac{562791312666017539}{876693583206100846}$

Table 3. Some general continued fractions converted to simple continued fractions

Q	R	C_n (dec.form)	C_n (Simple continued fraction sequence)
S_4	S_5	1.0821667605141670276 8 (corresponding to 6 terms)	1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1, 1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2 (39 terms)
S_5	S_4	1.04753443663236268392 (corresponding to 5 terms)	1,21,26,1,3,26,10,4,4,19,1,2,2,1,8,8,1,2,3,1, 10,1,2,1,2,3,1,4,1,8 (29 terms)
CS1	NCS1	1.54088941088788795255 (corresponding to 9 terms)	1,1,15,1,1,1,1,2,4,17,1,1,3,13,4,2,2,2,5,1,6,2, 2,9,2,15,1.51 (28 terms)

These sequences show no special regularities. As can be seen from table 3 the number of terms required to reach 20 decimals is much larger than for the corresponding general continued fractions.

A number of Smarandache periodic sequences were explored in the author's book *Computer Analysis of Number Sequences* [6]. An interesting property of simple continued fractions is that

> *A periodic continued fraction is a quadratic surd, i.e. an irrational root of a quadratic equation with integral coefficients.*

In terms of A_n and B_n, which for simple continued fractions are defined through

$$A_0=q_0,\quad A_1=q_0q_1+1,\qquad A_n=q_nA_{n-1}+A_{n-2}$$

$$B_0=1,\quad B_1=q_1,\qquad B_n=q_nB_{n-1}+B_{n-2} \tag{15}$$

the quadratic surd is found from the quadratic equation

$$B_nx^2+(B_{n-1}-A_n)x-A_{n-1}=0 \tag{16}$$

where n is the index of the last term in the periodic sequence. The relevant quadratic surd is

$$x=\frac{A_n-B_{n-1}+\sqrt{A_n^2+B_{n-1}^2-2A_nB_{n-1}-4A_{n-1}B_n}}{2B_n} \tag{17}$$

An example has been chosen from each of the following types of Smarandache periodic sequences:

1. The Smarandache two-digit periodic sequence:

Definition: Let N_k be an integer of at most two digits. N_k' is defined through

$$N_k' = \begin{cases} \text{the reverse of } N_k \text{ if } N_k \text{ is a two digit integer} \\ N_k\cdot 10 \text{ if } N_k \text{ is a one digit integer} \end{cases}$$

N_{k+1} is then determined by

$$N_{k+1}=|N_k-N_k'|$$

The sequence is initiated by an arbitrary two digit integer N_1 with unequal digits.

One such sequence is Q={9, 81, 63, 27, 45}. The corresponding quadratic equation is

$$6210109x^2-55829745x-1242703=0$$

2. The Smarandache Multiplication Periodic Sequence:

Definition: Let c>1 be a fixed integer and N_0 and arbitrary positive integer. N_{k+1} is derived from N_k by multiplying each digit x of N_k by c retaining only the last digit of the product cx to become the corresponding digit of N_{k+1}.

For c=3 we have the sequence Q={1, 3, 9, 7} with the corresponding quadratic equation

$$199x^2-235x-37=0$$

3. The Smarandache Mixed Composition Periodic Sequence:

Definition. Let N_0 be a two-digit integer $a_1\cdot 10+a_0$. If $a_1+a_0<10$ then $b_1=a_1+a_0$ otherwise $b_1=a_1+a_0+1$. $b_0=|a_1-a_0|$. We define $N_1=b_1\cdot 10+b_0$. N_{k+1} is derived from N_k in the same way.

One of these sequences is Q={18, 97, 72, 95, 54, 91} with the quadratic equation

$$3262583515x^2-58724288064x-645584400=0$$

and the relevant quadratic surd

$$x=\frac{58724288064+\sqrt{3456967100707577532096}}{6525167030}$$

The above experiments were carried out with *UBASIC* programs. An interesting aspect of this was to check the correctness by converting the quadratic surd back to the periodic sequence.

There are many interesting calculations to carry out in this area. However, this study will finish by this equality between a general continued fraction convergent and a simple continued fraction convergent.

[1,12,123,1234,12345,123456,1234567,*1,21,321,4321,54321,654321*]=
[1,12,5,1,6,1,1,1,48,7,2,1,20,2,1,5,1,2,1,1,9,1, 1,10,1,1,7,1,3,1,7,2,1,3,31,1,2,6,38,2]

References

1. J. Castillo, *Smarandache Continued Fractions*, Smarandache Notions Journal, Vol. 9, No. 1-2.
2. O. Perron, *Die Lehre von den Kettenbrüchen*, B.G. Teubner Verlagsgesellschaft, Stuttgart, 1954.
3. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, Fifth edition 1979.
4. H. Davenport, *The Higher Arithmetic*, Reprint: Dover Publications, Inc., New York, 1983.
5. C. Dumitrescu and V. Seleacu, *Some Notions and Questions in Number Theory, Vol. II*, Erhus University Press.
6. H. Ibstedt, *Computer Analysis of Number Sequences*, American Research Press, 1998.

SOME CONNECTIONS BETWEEN THE SMARANDACHE FUNCTION AND THE FIBONACCI SEQUENCE

Constantin Dumitrescu and Carmen Rocsoreanu

University of Craiova, Dept. of Mathematics
Craiova 1100, Romania

I. INTRODUCTION

The Smarandache function $S : N^* \to N^*$ is defined [9] by the condition that $S(n)$ is the smallest positive integer k such that $k!$ is divisible by n.

If

$$n = p_1^{\alpha_1} . p_2^{\alpha_{21}} . \dots . p_t^{\alpha_t} \tag{1}$$

is the decomposition of the positive integer n into primes, then it is easy to verify that

$$S(n) = \max\left(S\left(p_i^{\alpha_i}\right)\right) \tag{2}$$

One of the most important properties of this function is that a positive integer p is a fixed point of S if and only if p is a prime or $p = 4$.

This paper is aimed to provide generalizations of the Smarandache function. They will be constructed by means of sequences more general than the sequence of the factorials. Such sequences are monotonously convergent to zero sequences and divisibility sequences (in particular the Fibonacci sequence).

Our main result states that the Smarandache generalized function associated with every strong divisibility sequence (sequence satisfying the condition (15) from bellow) is a dual strong divisibility sequence (i.e. it satisfies the condition (26), the dual of (15)).

Note that the Smarandache function S is not monotonous. Indeed, $n_1 \le n_2$ does not imply $S(n_1) \le S(n_2)$. For instance $5 \le 12$ and $S(5) = 5$, $S(12) = 4$.

Let us denote by $\overset{d}{\vee}$ the least common multiple, by $\underset{d}{\wedge}$ the greatest common divisor and let $\wedge = \min$, $\vee = \max$. It is known that

$$N_0 = \left(N^*, \wedge, \vee\right) \text{ and } N_d = \left(N^*, \underset{d}{\wedge}, \overset{d}{\vee}\right)$$

are lattices. The order on N* corresponding to the lattice N_0 is the usual order:

$$n_1 \le n_2 \Leftrightarrow n_1 \wedge n_2 = n_1$$

and it is a total order. On the contrary, the order $\underset{d}{\le}$ corresponding to the lattice N_d, defined as

$$n_1 \underset{d}{\le} n_2 \Leftrightarrow n_1 \underset{d}{\wedge} n_2 = n_1$$

(the divisibility relation) is only a partial order.

More precisely we have

$n_1 \underset{d}{\leq} n_2 \Leftrightarrow n_1$ *divides* n_2.

For $n_1 \underset{d}{\leq} n_2$ we shall also write $n_2 \underset{d}{\geq} n_1$. We notice that N_d has *zero* as the greatest element, N_0 does not possess a greatest element and both lattices have 1 as the smallest element. Then it is convenient to consider in N_0 the convergence to infinity and in N_d, the convergence to zero.

Let

$$n_1 = \prod p_i^{\alpha_i} \text{ and } n_2 = \prod p_i^{\beta_i}$$

be the decompositions into primes of n_1 and n_2. Then we have

$$n_1 \overset{d}{\vee} n_2 = \prod p_i^{\max(\alpha_i, \beta_i)}.$$

The definition of the Smarandache function implies that

$$S\left(n_1 \overset{d}{\vee} n_2\right) = S(n_1) \vee S(n_2) \quad (3)$$

Also we have

$$n_1 \underset{d}{\leq} n_2 \Rightarrow S(n_1) \leq S(n_2). \quad (4)$$

In order to make explicit the lattice (so, the order) on the set N^*, we shall write N_0 instead of N^*, if the order on the set of the positive integers is the usual order and N_d instead of N^*, if we consider the order $\underset{d}{\leq}$ respectively.

Then (4) shows that the Smarandache function, considered as a function

$$S : N_d \to N_0, \quad (5)$$

is an order preserving map.

From (2) it follows that the determination of $S(n)$ reduces to the computation of $S(p^\alpha)$. In addition, it is proved [1] that if the sequence

$$(p): \ 1,\ p,\ p^2,\ \ldots,\ p^i,\ \ldots \quad (6)$$

is the standard $p-scale$ and the sequence

$$[p]: \ a_1(p),\ a_2(p),\ \ldots,\ a_i(p),\ \ldots$$

is the generalized numerical scale determined by the sequence

$$a_i(p) = \frac{p^i - 1}{p - 1}$$

then

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \quad . \quad (7)$$

In other words, $S(p^\alpha)$ can be obtained by multiplying by p the number obtained writing the exponent α in the generalized scale $[p]$ and "reading" it in the scale (p).

For instance, in order to calculate $S(3^{99})$ let us consider the scale

[3] 1, 4, 13, 40, 121, ...

Then, for $\alpha = 99$, we have

$$\alpha_{[3]} = 2a_4(3) + a_3(3) + a_2(3) + 2a_1(3) = 2112_{[3]}$$

and "reading" this number in the usual scale

(3) $\quad 1,\ 3,\ 3^2,\ 3^3,\ \ldots$

we get $S(3^{99})=3(2\cdot 3^3+3^2+3+2)$=204. So, 204 is the smallest positive integer whose factorial is divisible by 3^{99}.

We quote also the following formula used to compute $S(p^\alpha)$:

$$S(p^\alpha)=(p-1)\alpha+\sigma_{[p]}(\alpha), \tag{8}$$

where $\sigma_{[p]}(\alpha)$ stands for the sum of the digits of the integer α written in the scale $[p]$.

2. GENERALIZED SMARANDACHE FUNCTIONS

A scquence of positive integers is a mapping $\sigma : N^* \to N^*$ and it is usualy denoted by $(\sigma_n)_{n\in N^*}$ (i.e. the set of its values). Since in the sequel an essential point is to make cvident the structure (the lattice) on the domain and on the range of this function respectively, we adopt the notation from (5).

Then

$$\sigma : N_0 \to N_d \tag{9}$$

shows that σ is a sequence of positive integers defined on the set N^*. This set was structured as a lattice by $\wedge$ and $\vee$ and its range has also a structure of lattice, induced by $\underset{d}{\wedge}$ and $\overset{d}{\vee}$.

Definition 2.1. [3] The sequence (9) is a *multiplicatively convergent to zero* sequence (mcz) if

$$(\forall) n\in N^* \quad (\exists)\quad m_n \in N^* \quad (\forall) m\geq m_n \Rightarrow n \underset{d}{\leq} \sigma(m). \tag{10}$$

In other words, a (mcz) sequence is a sequence defined as in (9), which is *convergent to zero*.

These sequences, satisfying in addition the condition

$$\sigma(n)\underset{d}{\leq}\sigma(n+1) \tag{11}$$

(that is $\sigma(n)$divides $\sigma(n+1)$) were considered by G. Christol [3] in order to obtain a generalization of $p-adic$ numbers.

As an example of a (mcz) sequence we may consider the sequence defined by $\sigma(n)=n!$. This sequence also satisfies the condition (11).

Remark 2.1. We find that the value $S(n)$ of tbe Smarandache function at the point n is the smallest integer m_n provided by (10), whenever $\sigma(n)=n!$. This enables us to define a Smarandache type function for each (mcz) sequence. Indeed, for an arbitrary (mcz) sequence σ, we may define $S_\sigma(n)$ as the smallest integer m_n given by (10).

The (mcz) sequences satisfying the extra-condition (11) generalize thc factorial. Indeed, if

$$\sigma(n+1)=k_{n+1}\sigma(n) \qquad (12)$$

then

$\sigma(n)=k_1 \cdot k_2 \cdot \ldots \cdot k_n$, with $k_1 = 1$ and $k_i \in N^*$ for $i>1$.

Starting with the lattices N_0 and N_d, we can construct sequences

$$\sigma : N_d \to N_d \ . \qquad (13)$$

Definition 2.2. A sequence (13) is called a *divisibility sequence (ds)* if

$$n \underset{d}{\leq} m \Rightarrow \sigma(n) \underset{d}{\leq} \sigma(m) \qquad (14)$$

(that is if the mapping σ from (13) ia monotonous). The sequence (13) is called a *strong divisibility sequence (sds)* if

$$\sigma\left(n \underset{d}{\wedge} m\right)=\sigma(n) \underset{d}{\wedge} \sigma(m) \text{ for every } n, m \in N^* . \qquad (15)$$

Strong divisibility sequences are considered, for instance, by N. Jensen in [5].

It is known that the Fibonacci sequence is also (sds).

For a sequence σ of positive integers, concepts as (usual) monotonicity, multiplicatively convergence to zero, divisibility, have been independently studied by many authors. A unifying treatement of these concepts can be achieved if we remark that they are monotonicity or convergence conditions of a given sequence $\sigma : N^* \to N^*$, for adequate lattices on N^*.

We shall consider now all the possibilities to define a sequence of positive integers, with respect to the lattices N_0 and N_d. To make briefly evident thc kind of the lattice considered on the domain and on the range of α, we shall use the following notation:

(a) *a sequence* $\sigma_{oo} : N_o \to N_o$ *is an* (oo)*- sequence*

(b) *a sequence* $\sigma_{od} : N_o \to N_d$ *is an* (od)*- sequence*

(c) *a sequence* $\sigma_{do} : N_d \to N_o$ *is an* (do)*- sequence*

(d) *a sequence* $\sigma_{dd} : N_d \to N_d$ *is a* (dd)*- sequence*

We have already seen (Remark 2.1) that, considering (mcz) sequences, the Smarandache function may be generalized.

In order to generalize tbe Smarandache function for each type of the above sequences, it is necessary to consider the monotonicity and the existence of a limit corresponding to each of the cases (a) - (d).

Of course, the limit is *infinit* for N_o-valued sequence and it is *zero* for the others. We have four kinds of monotonicity.

For a (do)*- squence* σ_{do}, the monotonicity reads:

$$(m_{do}) \quad (\forall) n_1, n_2 \in N^*, \quad n_1 \underset{d}{\leq} n_2 \Rightarrow \sigma_{do}(n_1) \leq \sigma_{do}(n_2)$$

and the condition of convergence to infinity is:

$$(c_{do}) \quad (\forall) n \in N^* \quad (\exists) m_n \in N^* \quad (\forall) m \underset{d}{\geq} m_n \Rightarrow \sigma_{do}(m) \geq n .$$

Similarly, for a (dd)*- sequence* σ_{dd}, the monotonicity reads:

$$(m_{dd})\quad (\forall)n_1, n_2 \in N^*, \quad n_1 \underset{d}{\le} n_2 \Rightarrow \sigma_{dd}(n_1) \underset{d}{\le} \sigma_{dd}(n_2)$$

and the convergence to zero is:

$$(c_{dd})\quad (\forall)n \in N^* \quad (\exists)m_n \in N^* \quad (\forall)m \underset{d}{\ge} m_n \Rightarrow \sigma_{dd}(m) \underset{d}{\ge} n.$$

Definition 2.3. The *generalized Smarandache function* associated to a sequence σ_{ij} satisfying tbe condition (c_{ij}), with $i, j \in \{o, d\}$, is

$$S_{ij}(n) = \min\{m_n \mid m_n \textit{ given by the condition } (c_{ij})\} \tag{16}$$

Remark that $(oo)-$*sequences* are the classical sequences of positive integers. As examples of $(od)-$*sequences* we quote the (mcz) sequences. Examples of $(dd)-$*sequences* are (ds) and $(sds)-$*sequences*. Finally, the generalized Smarandache functions S_{od} associated with $(od)-$*sequences* satisfying the condition (c_{od}) are $(do)-$*sequences*.

The functions S_{ij} have the following properties:

Theorem 2.1. *Every function S_{oo} satisfies:*

$(i)\quad (\forall)n_1, n_2 \in N^*, \quad n_1 \le n_2 \Rightarrow S_{oo}(n_1) \le S_{oo}(n_2),$

that is S_{oo} satisfies (m_{oo}).

$(ii)\quad S_{oo}(n_1 \vee n_2) = S_{oo}(n_1) \vee S_{oo}(n_2)$

$(iii)\quad S_{oo}(n_1 \wedge n_2) = S_{oo}(n_1) \wedge S_{oo}(n_2).$

Proof: (i) The definition of $S_{oo}(n)$ implies that:

$S_{oo}(n_i) = \min\{m_{n_i} \mid (\forall) m \ge m_{n_i} \Rightarrow \sigma_{oo}(m) \ge n_i\}$, for $i = 1, 2$

Therefore

$$(\forall)m \ge S_{oo}(n_2) \Rightarrow \sigma_{oo}(m) \ge n_2 \ge n_1$$

and so $S_{oo}(n_1) \le S_{oo}(n_2)$. The equalities (ii) and (iii) are consequences of (i).

Theorem 2.2. *Every function S_{od} has the following properties:*

$(iv)\quad (\forall)n_1, n_2 \in N^*, \quad n_1 \underset{d}{\le} n_2 \Rightarrow S_{od}(n_1) \le S_{od}(n_2)$

that is S_{od} satisfies (m_{od}).

$(v)\quad S_{od}\left(n_1 \overset{d}{\vee} n_2\right) = S_{od}(n_1) \vee S_{od}(n_2).$

$(vi)\quad S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \le S_{od}(n_1) \wedge S_{od}(n_2).$

Proof: The equality (v) may be proved in the same manner as the equality (3) for the function S. Then from (v) it follows (iv).

For (vi) let us note $u = S_{od}(n_1) \wedge S_{od}(n_2)$. From

$$n_1 \underset{d}{\wedge} n_2 \underset{d}{\le} n_1, \quad n_1 \underset{d}{\wedge} n_2 \underset{d}{\le} n_2$$

and from (iv), it follows that

$$S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{od}(n_1), \quad S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{od}(n_2),$$

so $S_{od}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{od}(n_1) \wedge S_{od}(n_2)$.

Theorem 2.3. *The functions S_{do} satisfy*:

(vii) $(\forall) n_1, n_2 \in N^*, \quad n_1 \leq n_2 \Rightarrow S_{do}(n_1) \leq S_{do}(n_2)$.

$(viii)$ $S_{do}(n_1 \vee n_2) \leq S_{do}(n_1) \overset{d}{\vee} S_{do}(n_2)$.

(ix) $S_{do}(n_1 \vee n_2) = S_{do}(n_1) \vee S_{do}(n_2)$.

(x) $S_{do}(n_1 \wedge n_2) = S_{do}(n_1) \wedge S_{do}(n_2)$.

Proof: Let us note that (ix) and (x) are consequences of (vii). In our terms (vii) is just the fact that the Smarandache generalized function S_{do} associated with a $(do)-$*sequence* is $(oo)-$*monotonous*. To prove this assertion, let $n_1 \leq n_2$. Then for every $m \underset{d}{\geq} m_{n_2}$, we have

$$\sigma_{do}(m) \geq n_2 \geq n_1$$

and so $S_{do}(n_1) \leq S_{do}(n_2)$.

$(viii)$ For i = 1, 2 we have:

$$S_{do}(n_i) = \min \left\{ m_{n_i} \,|\, (\forall) m \underset{d}{\geq} m_{n_i} \Rightarrow \sigma_{do}(m) \geq n_i \right\}$$

Let us suppose that $n_1 \leq n_2$, so $n_1 \vee n_2 = n_2$ and $S_{do}(n_1 \vee n_2) = S_{do}(n_2)$. If we take $m_0 = S_{do}(n_1) \overset{d}{\vee} S_{do}(n_2)$, then for every $m \underset{d}{\geq} m_0$ it follows that $\sigma_{do}(m) \geq n_i$, for i = 1, 2, so $\sigma_{do}(m) \geq n_1 \vee n$, whence the desired inequality.

Consequence 2.1. $S_{do}(n_1) \underset{d}{\wedge} S_{do}(n_2) \leq S_{do}(n_1) \wedge S_{do}(n_2) = S_{do}(n_1 \wedge n_2) \leq$

$S_{do}(n_1) \vee S_{do}(n_2) = S_{do}(n_1 \vee n_2) \leq S_{do}(n_1) \overset{d}{\vee} S_{do}(n_2)$.

Theorem 2.4. *The functions S_{dd} satisfy*:

(xi) $S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) \leq S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2)$.

(xii) *If $n_1 \underset{d}{\leq} n_2$ or $n_2 \underset{d}{\leq} n_1$ then*

$$S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) = S_{dd}(n_1) \vee S_{dd}(n_2).$$

$(xiii)$ $S_{dd}\left(n_1 \underset{d}{\wedge} n_2\right) \leq S_{dd}(n_1) \wedge S_{dd}(n_2)$.

Proof : The proof of (xi) is similar to the proof of $(viii)$ and the other assertions may be easily obtained by using the definition of S_{dd} from (17) (for $i = j = d$).

Consequence 2.2. For all $n_1, n_2 \in N^*$ we have

$$S_{dd}(n_1) \vee S_{dd}(n_2) \le S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) \le S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2).$$

This follows from the fact that

$$n_i \underset{d}{\le} n_1 \overset{d}{\vee} n_2 \quad for\ i = 1, 2 \Rightarrow S_{dd}(n_i) \le S_{dd}\left(n_1 \overset{d}{\vee} n_2\right).$$

If σ_{dd} is a divisibility sequence, the above theorem implies that the associated Smarandache function satisfies the inequality (xi). In the following we shall see that, if the sequence σ_{dd} is a divisibility sequence with additional properties, namely if it is a strong divisibility sequence, then the inequality (xi) becomes equality.

Theorem 2.5: *If σ_{dd} is a (sds) satisfying the condition (c_{dd}), then*:

$$S_{dd}\left(n_1 \overset{d}{\vee} n_2\right) = S_{dd}(n_1) \overset{d}{\vee} S_{dd}(n_2) \tag{17}$$

and

$$(\forall) n_1, n_2 \in N^*, \quad n_1 \underset{d}{\le} n_2 \Rightarrow S_{dd}(n_1) \underset{d}{\le} S_{dd}(n_2) \tag{18}$$

(*i.e.* S_{dd} *satisfies the monotonicity condition* (m_{dd})).

Proof: In order to prove the equality (17), it is sufficient to show that

$$S_{dd}(n_i) \underset{d}{\le} S_{dd}\left(n_1 \overset{d}{\vee} n_2\right), \quad \text{for } i = 1, 2.$$

But if, for instance, the above inequality does not hold for n_1 and we denote

$$d_O = S_{dd}(n_1) \underset{d}{\wedge} S_{dd}\left(n_1 \overset{d}{\vee} n_2\right),$$

it follows that $d_O < S_{dd}(n_1)$ and taking into account that

$$\sigma_{dd}(S_{dd}(n_1)) \underset{d}{\ge} n_1 \quad and \quad n_1 \underset{d}{\le} n_1 \overset{d}{\vee} n_2 \underset{d}{\le} \sigma_{dd}\left(S_{dd}\left(n_1 \overset{d}{\vee} n_2\right)\right),$$

we have

$$\sigma_{dd}(d_O) = \sigma_{dd}\left(S_{dd}(n_1) \underset{d}{\wedge} S_{dd}\left(n_1 \overset{d}{\vee} n_2\right)\right) =$$

$$= \sigma_{dd}(S_{dd}(n_1)) \underset{d}{\wedge} \sigma_{dd}\left(S_{dd}\left(n_1 \overset{d}{\vee} n_2\right)\right) \underset{d}{\ge} n_1 \underset{d}{\wedge} n_1 = n_1.$$

Thus, we obtain the contradiction

$$S_{dd}(n_1) \le d_O < S_{dd}(n_1).$$

So, if the sequence σ_{dd} is a (sds), that is if the equality (15) holds, then the corresponding Smarandache function S_{dd} satisfies the dual equality (17).

Example. The Fibonacci sequence $(F_n)_{n \in N^*}$ is a (sds). Therefore, the generalized Smarandache function S_F associated with this sequence satisfy:

$$S_F\left(n_1 \overset{d}{\vee} n_2\right) = S_F(n_1) \overset{d}{\vee} S_F(n_2) \tag{19}$$

By means of this equality, the computation of $S_F(n)$ reduces to the determination of $S_F(p^\alpha)$, where p is a prime number. For instance

$S_F(52) = \min\left\{m_n \mid (\forall) m \underset{d}{\geq} m_n \Rightarrow 52 \underset{d}{\leq} F(m)\right\} =$

$= S_F(2^2) \overset{d}{\vee} S_F(13) = 6 \overset{d}{\vee} 7 = 42$.

So, 42 is the smallest positive integer m such that $F(m)$ is divisible by 52.

Also, we have

$$S_F(12) = S_F(2^2 \cdot 3) = S_F(2^2) \overset{d}{\vee} S_F(3) = 6 \overset{d}{\vee} 4 = 12, \tag{20}$$

therefore $n = 12$ is a fixed point of S_F.

The values of $S_F(p^\alpha)$ may be obtained by writing all F_n in the scale (p) given by (6), which is a difficult operation. At the time being, we are not able to provide a closed formula for the computation of $S_F(p^\alpha)$. However, we shall present some partial results in this direction. In [8] it is stated that

$3^k \underset{d}{\leq} F_n \Leftrightarrow 4 \cdot 3^{k-1} \underset{d}{\leq} n$

$2^k \underset{d}{\leq} F_n \Leftrightarrow 3 \cdot 2^{k-2} \underset{d}{\leq} n, \quad for\ k \geq 3$.

It is known (see for instance [6], [7]) that if σ is a non-degenerate second-order linear recurrence sequence defined by

$$\sigma(n) = A\sigma(n-1) - B\sigma(n-2) \tag{21}$$

where A and B are fixed non-zero coprime integers and $\sigma(1) = 1,\ \sigma(2) = A$, then

$$n \in Z^*, \quad n \underset{d}{\wedge} B = 1 \Rightarrow (\exists) m \in N^* \quad n \underset{d}{\leq} \sigma(m). \tag{22}$$

The least index of these terms is called the rank of appearance of n in the sequence and is denoted by $r(n)$.

If $D = A^2 - 4B$ and (D/n) stands for the Jacobi symbol, then for $mn \underset{d}{\wedge} BD = 1$ and p a prime we have ([6])

$$\begin{aligned} &n \underset{d}{\leq} \sigma(m) \Leftrightarrow r(n) \underset{d}{\leq} m; \quad r(p) \underset{d}{\leq} p - (D/p) \\ &r(p) \underset{d}{\leq} \frac{p-(D/p)}{2} \Leftrightarrow (B/p) = 1; \quad r\left(m \overset{d}{\vee} n\right) = r(m) \overset{d}{\vee} r(n). \end{aligned} \tag{23}$$

Let us denote $N_B^* = \left\{n \in N^* \mid n \underset{d}{\wedge} B = 1\right\}$. Obviously, if r is considered as a function $r: N_B^* \to N^*$, then we can write:

$r(n) = \min\left\{m \mid n \underset{d}{\leq} \sigma(m)\right\}$.

Whence an evident parallel between the above methods described for the construction of the generalized Smarandache functions and the definition of the function r.

For the Fibonacci sequence (F_n) we have $A = 1,\ B = -1$ and so $D = 5$.

This implies

$$p = 5k \pm 1 \Rightarrow (5/p) = 1 \tag{24}$$

$$p = 5k \pm 2 \Rightarrow (5/p) = -1 \tag{25}$$

and it follows that if (24) holds, then p divides F_{p-1}. Thus $S_F(p)$ is a divisor of p - 1. In the second case p divides F_{p+1} and $S_F(p)$ is a divisor of $p + 1$.

From (23) we deduce

$$S_F(p) \leq p - (5/p)$$

for any prime number p.

Lemma 2 from [6] implies that the fraction $(p-(5/p))/S_F(p)$ is unbounded. We also have

$$p^k \underset{d}{\leq} F_n \Leftrightarrow S_F(p^k) \underset{d}{\leq} n.$$

Example. For $p = 11$ it follows $(5/p) = 1$, so $S_F(11) \underset{d}{\leq} 10$. In fact, we have precisely $S_F(11) = 11 - (5/11) = 10$, but there exist prime numbers such that $S_F(p) < p - (5/p)$. For instance, $p = 17$, for which $p - (5/p) = 18$ and $S_F(17) = 9$.

Definition 2.4. The sequence σ is *a dual strong divisibility sequence* (*dsds*) if

$$\sigma\left(n \overset{d}{\vee} m\right) = \sigma(n) \overset{d}{\vee} \sigma(m) \quad \textit{for all } n, m \in N^*. \tag{26}$$

It may be easily seen that every strong divisibility sequence is a divisibility sequence. We also have:

Proposition 2.1 *Every dual strong divisibility sequence is a divisibility sequence.*

Proof. We have to prove that (26) implies (14). But if $n \underset{d}{\leq} m$, it follows $n \overset{d}{\vee} m = m$ and then

$$\sigma(m) = \sigma\left(n \overset{d}{\vee} m\right) = \sigma(n) \overset{d}{\vee} \sigma(m) \tag{27}$$

so, $\sigma(n) \underset{d}{\leq} \sigma(m)$.

Then Theorem *2.5* asserts that the Smarandache generalized function S_σ associated with any strong divisibility sequence σ is a dual strong divisibility sequence. Of course, in this case, both sequences σ and S_σ are divisibility sequences.

It would be very interesting to prove whether the converse assertion holds. That is if S_{dd} is the generalized Smarandache function associated with a (divisibility) sequence σ_{dd} satisfying the condition (c_{dd}), then the equality (17) implies the strong divisibility.

Remarks. (1) It is known that the Smarandache function S is *onto*. But given a $(dd)-$*sequence* σ_{dd}, even if it is a (*sds*), it does not follow that the associated function S_{dd} is *onto*. Indeed, the function S_F associated with the Fibonacci sequence is not *onto*, because $n = 2$ is not a value of S_F.

(2) One of the most interesting diophantine equations associated with a Smarandache type function is that which provides its fixed points. We remember that the fixed points for the Smarandarche function are all the primes and the composit number $n = 4$. For the functions S_{dd} the equation providing the fixed points reads $S_{dd}(x) = x$ and for S_F we have as solutions, for instance, $n = 5$, $n = 12$.

At the end of this paper we quote the following question on the Smarandache function, also related to the Fibonacci sequence:

T. Yau [10] wondered if there exist triplets of positive integers (n, n-1, n-2) such that the corresponding values of the Smarandache function satisfy the Fibonacci recurrence relation $S(n) = S(n-1) + S(n-2)$.

He found two such triplets, namely for $n = 11$ and for $n = 121$. Indeed, we have $S(9) + S(10) = S(11)$ and $S(119) + S(120) = S(121)$.

Using a computer, Charles Ashbacher [2] found additional values. These are for $n = 4902$, $n = 26245$, $n = 32112$, $n = 64010$, $n = 368139$, $n = 415664$.

Recently H. Ibsent [4] proposed an algorithm permitting to find, by means of a computer, much more values. But the question posed by T. Yau "How many other triplets with the same property exist?"is still unsolved.

Acknowledgements

We wish to express our gratitude to Professor Adelina Georgescu for her important contribution to the English translation of this paper.

REFERENCES

1. M. Andrei, C. Dumitrescu, V. Seleacu, L. Tutescu, St. Zanfir: "Some Remarks on the Smarandache Function", *Smarandache Function Journal*, Vol. 4-5, No. 1, (1994), pp. 1-5.
2. Ashbacher, Ch.: " Some Problems on Smarandache Function ", *Smarandache Function Journal*, Vol. 6, No. 1, (1995), pp. 21-36.
3. Christol, G.: "p-adic Numbers and Ultrametricity", *From Number Theory to Phisics*, M. Woldschmidt et all Editors, Springer-Verlag, 1992, pp. 440-475.
4. Ibsent, H: "Bases Solutions (The Smarandache Function)", *Smarandache Notion Journal*, Vol. 7, No. 1, (1996),86-89.
5. Jensen, N.: "Some Results on Divisibility Sequences", *Appl. of Fibonacci Numbers*, Vol. 3, G. E. Bergum et al. Editors, Kluwer Academic Press, 1990, pp. 181-189.
6. Kiss P., Phong B. M., Lieuwens E.: "On Lucas Pseudoprimes Which Are Products of s Primes", *Fibonacci Numbers and Their Applications*, Vol. 1, *A. N. Philippou et al. Editors*, *D. Reidel Publishing Company*, 1986, pp. 131-139.
7. Kiss, P.: "Primitive Divisors of Lucas Numbers" *Appl. of Fibonacci Numbers*, Vol. 2, A. N. Philippou et al. Editors, Kluwer Academic Prees, 1988, pp. 29-38.
8. Robbins N.: "Fibonacci Numbers of the Forms $PX^2 \pm 1$, $PX^3 \pm 1$, Where P is a Prime", *Appl. of Fibonacci Numbers*, Vol. 2, A. N. Philippou et al. Editors, Kluwer Academic Press, 1988, pp. 77-88.
9. Smarandache, F.: "A Function in the Number Theory", *An. Univ. Timisoara*, Ser. St. Mat. Vol. XVIII, fasc. 1, (1980), pp. 79-88.

10. Yau, T.: "A Problem Concerning the Fibonacci Series", *Smarandache Function Journal*, Vol. 4-5, No. l, (1984), pp. 42.

AMS Classification Numbers: 11 A 25, 11B39.

COMPUTATIVE PARADOXES IN MODERN DATA ANALYSIS

Y. V. CHEBRAKOV AND V. V. SHMAGIN
Department of Mathematics, Technical University,
Nevsky 3-11, 191186, St-Petersburg, Russia
E-mail: chebra@phdeg.hop.stu.neva.ru

By developing F. Smarandache thema on paradoxes in mathematics it is stated, firstly, if in measurement (natural science) experiments the best solutions are found by using methods of modern data analysis theory, then some difficulties with the interpretation of the computation results are liable to occur; secondly, one is not capable to overcome these difficulties without a data analysis theory modification, consisted in the translation of this theory from Aristotelian "binary logic" into more progressive "fuzzy logic".

Key words: data analysis, revealing outliers, confidence interval, fuzzy logic.

1 Introduction

As generally known from history of science, a scientific theory may have crisis in process of its development, when it disjoints in a set of fragment theories, that weak-coordinate each other and, as a whole, form a collection of various non-integrated conceptions. For instance, as we assume, F. Smarandache mathematical notions and questions [1-2] help us to understand quite well that a stable equilibrium, observed in mathematics at the present time, is no more than fantasy. Thus, it falls in exactly with F. Smarandache views that the finding and investigating paradoxes in mathematics is a very effective way of approximating to the truth and so at present each of scientific researches, continuing F. Smarandache thema[2], should be considered as very actual one.

Let us assume that *computative paradoxes* in mathematics are mainly such computation results, obtained by using mathematical methods, which are contradicted some mathematical statements. The main goal of this paper is to demonstrate that the mentioned crisis, demanding practical action instead of debate, occurs in modern data analysis, which formally has its own developed mathematical theory, but does not capable "to cope worthily" with a large number of practical problems of quantitative processing results of measurement experiments.

Another goal of this paper is to equip the mathematicians and software designers, working in the data analysis field, with a set of examples, demonstrating dramatically that, if, for solving some problems on analysing data arrays, one uses the standard computer programmes and/or time-tested methods of modern data analysis theory, then a set of the paradoxical computative results may be obtained.

2 Approximative problems of data analysis

2.1 The main problems of regression analysis theory and standard solution methods

As generally known[3-7], for found experimental dependence $\{y_n, x_n\}$ ($n = 1, 2, \ldots, N$) and given approximative function $F(\mathbf{A}, x)$, in the measurement (natural science) experiments the main problems of regression analysis theory are finding estimates of $\mathbf{A}'$ and y' and variances of $\delta\mathbf{A}'$ and $\delta(y-y')$, where $\mathbf{A}'$ is an estimate of vector parameter $\mathbf{A}$ of the function $F(\mathbf{A}, x)$ and $\{y_n'\} = \{F(\mathbf{A}', x_n)\}$. In particular, if $F(\mathbf{A}, x) = \sum_{l=1}^{L} a_l h_l(x)$ $\{F(\mathbf{A}, x)$ is a *linear* model$\}$, where $h_l(x)$ are some functions on x, then in received regression analysis theory *standard* solution of discussed problems has form

$$\mathbf{A}' = (\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\,\mathbf{H}^{\mathrm{T}}\mathbf{Y}, \quad (\delta\mathbf{A}')^2 = s/(N-L)\ \mathrm{diag}(\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1} \qquad (1)$$

$$\delta_p(y-y') = y' \pm t_p\, s\sqrt{1 + \mathbf{H}_i^{\mathrm{T}}(\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}_i}\,,$$

where $\mathbf{H}$ is a matrix $L{\times}N$ in size with n-th row $(h_1(x_n), h_2(x_n), \ldots, h_L(x_n))$; $\mathbf{H}^{\mathrm{T}}$ is the transposed matrix $\mathbf{H}$; $\mathbf{Y} = \{y_n\}$; $s = \sum_{n=1}^{N}(y_n - y_n')^2 / (N-L)$; $\mathbf{H}_i = (h_1(x_i), h_2(x_i), \ldots, h_L(x_i))$; the value of t_p is determined by t-Student distribution table and generally depends on the assigned value of the significance level of p and the value of $N-L$ (a number of freedom degree); at the assigned value of the significance level of p the notation of $\delta_p(y-y')$ means confidence interval for possible deviations of experimental values of y from computed values $y' = F(\mathbf{A}', x_i)$. According to Gauss - Markov theorem[4, 5], for classical data analysis model

$$y_n = F(\mathbf{A}, x_n) + e_n \qquad (2)$$

the solution (1) is the best (gives minimum value of s), if the following conditions are fulfilled:

all values of $\{x_n\}$ are not random, mathematical expectation of random value $\{e_n\}$ is equal to zero and random values of $\{e_n\}$ are non-correlated and have the same dispersions σ^2.

Example 1. In table 1 we adduce an experimental data array, obtained by Russian chemist D.I.Mendeleev in 1881, when he investigated the solvability (y, relative units) of sodium nitrate ($NaNO_3$) on the water temperature (x, °C).

Table 1.

D.I.Mendeleev data array

n	x_n	y_n	$y_n - y_n'$	n	x_n	y_n	$y_n - y_n'$	n	x_n	y_n	$y_n - y_n'$
1	0	66.7	–0.80	4	15	80.6	0.05	7	36	99.4	0.58
2	4	71.0	0.02	5	21	85.7	–0.07	8	51	113.6	1.73
3	10	76.3	0.10	6	29	92.9	0.17	9	68	125.1	–1.56

By analysing the data array $\{y_n, x_n\}$, presented in table 1, Y.V.Linnik[3] states that these data, as it was noted by D.I.Mendeleev, are well-fitted by linear model $y' = 67.5 + 0.871x$ $(\delta A' = (0.5; 0.2))$, although the correspondence between experimental and computed on linear model values of y is slightly getting worse at the beginning and end of investigated temperature region (see the values $\{y_n - y'_n\}$ adduced in table 1). We add that for discussed data array Y.V.Linnik[3] computes the confidence interval of $\delta_p\,(y - y')$ from (1) at the significance level of $p = 0.9$:

$$\delta_{0.9}\,(y - y') = \pm\, 0.593\,\sqrt{1 + (x - 26)^2 / 4511} \,. \tag{3}$$

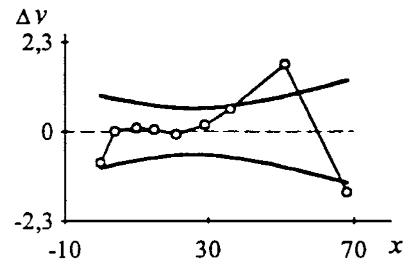


Figure 1. The plots of confidence interval of the deviation of y from y' (heavy lines) and residuals $y - y'$ (circles) for D.I.Mendeleev data array.

We show the plots of $\delta_{0.9}\,(y - y')$ on x by heavy lines in figure 1 and $\{y_n - y_n', x_n\}$ by the circles. Since the plot of $\{y_n - y_n', x_n\}$ steps over the heavy lines in figure 1, some computative difficulty is revealed:

the standard way (1), used by Y.V.Linnik[3] for determining the confidence interval of the deviations of y from y', is out of character with the discussed experimental data array.

It follows from results presented in table 1 and/or figure 1, if one assumes that $\delta(y - y') \geq \max |y_n - y_n'| = 1.73$ then the broken connections of the confidence interval $\delta(y - y')$ with D.I.Mendeleev data array will be pieced up. But values of $\delta A'$, calculated by Y.V.Linnik from (1), disagree with the values $\delta(y - y') \geq 1.73$, and, consequently,

standard values of δA' is out of character with D.I.Mendeleev data array also.

2.2 Alternative methods of regression analysis theory

P.Huber[8] noted that, as the rule, 5 – 10% of all observations in the majority of analysing experimental arrays are anomalous or, in other words, the conditions of Gauss - Markov theorem, adduced above, are not fulfilled. Consequently, in practice instead of the standard solution (1), found by "least squares (LS) method", *alternative* methods, developed in the frames of received regression analysis theory, should be used. In particular, if the data array $\{y_n, x_n\}$ contains a set of

outliers, then for finding the best solution of discussed problem it is necessary[6, 7] or to remove all outliers from the analysing data array (*strategy 1*), or to compute the values of $\mathbf{A}'$ on the initial data array by means of M-robust estimators (*strategy 2*). For revealing outliers in the data array P.J.Rousseeuw and A.M.Leroy[9] suggest to use one of two combined statistical procedures, in which parameter estimates, minimising the median of the array $\{(y_n - y_n')^2\}$ (*the first procedure*) or the sum of K first elements of the same array (*the second procedure*), are considered as the best ones. If $F(\mathbf{A}, x)$ is a linear function (see above), then the robust M-estimates of $\mathbf{A}'$ are obtained as result of the solving of one from two minimisation problems[6-9]

$$S_\varphi(\mathbf{A}) = \sum_{n=1}^{N} \varphi(y_n - y_n') \Rightarrow \min \quad \text{or} \quad \partial S_\varphi / \partial a_l = \sum_{n=1}^{N} \psi(y_n - y_n')\, h_l(x_n) = 0, \qquad (4)$$

where function $\varphi(r)$ is symmetric concerning Y-axis, continuously differentiable with a minimum at zero and $\varphi(0) = 0$; $\psi(r)$ is a derivative of $\varphi(r)$ with respect to r.

Continued example 1. Since D.I.Mendeleev data array from table 1 contains outliers, we adduce results of quantitative processing this data by alternative methods, defined above.

1. Let in (4) Andrews function[10] be applied: $\varphi(r) = d(1-\cos(r/d))$ if $|r| \le d\pi$ and $\varphi(r) = 0$ if $|r| > d\pi$. It is articulate in figure 2 that in this case the values of the linear model parameters a_0 and a_1 depend on

a) the values of parameter d of Andrews function $\varphi(r)$;

b) the type of the minimisation robust regression problem (solutions of the first and second minimisation problem of (4) are marked respectively by triangles and circles in figure 2).

Thus, in this case a computative paradox declares itself in the fact, that

in actual practice the robust estimates are not robust

and so, as K.R.Draper and H.Smith[11] wrote already,

"unreasoning application of robust estimators looks like reckless application of ridge-estimators: they can be useful, but can be improper also. The main problem is such one, that we do not know, which robust estimators and at which types of supposes about errors are effectual to applicate; but some investigations in this direction have been done..."

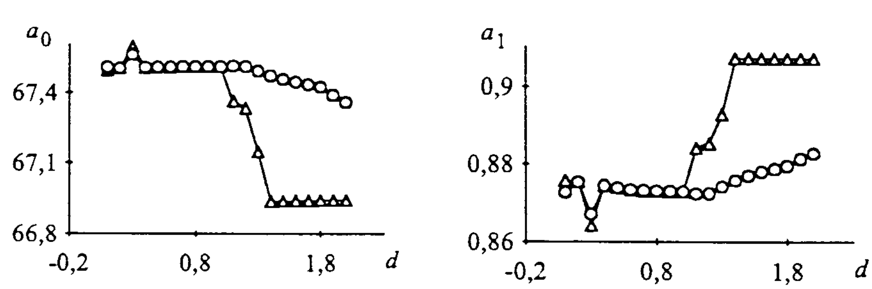


Figure 2. Dependences of parameters values of linear model a_0+a_1x on values internal parameter of robust Andrews estimator and the type of the minimisation problems (4).

2. Let us reveal outliers in D.I.Mendeleev data array by both combined statistical procedures [9], mentioned above.

Our computation results show

a) both procedures could not find the all four outliers (1, 7, 8 and 9) but the only three ones with numbers 1, 8 and 9;

b) if a set of readings with numbers 1, 8 and 9 is deleted from D.I.Mendeleev data array, then for the truncated data array the first procedure will not find a new outlier, but the second procedure will find two outliers yet that have numbers 2 and 6 in the initial data array.

Thus, in this case the main computative paradox is exhibited in the fact, that

revealing outliers problems solutions depend on a type of the used statistical procedures.

It remains for us to add, if one

a) computes y' by formula [6, 7]

$$y'(\xi,\ x) = g_\alpha(\ 68.12 + 0.02\xi + (0.85652 - 0.00046\xi)\, x\ ), \tag{5}$$

then for each n the difference $|\, y_n - y_n'\, |$ will keep within the limit of the chosen above value for the confidence interval $\delta(y - y')$, where α=0.94; $0 \le \xi \le 35$; $g_\alpha(y) = 2\alpha[y/(2\alpha)] + 2\alpha$ at $|\, y - 2\alpha\,[y/(2\alpha)]\,| \ge \alpha$, otherwise $g_\alpha(y) = 2\alpha\,[y/(2\alpha)]$, $[\,b\,]$ means integer part of b, Thus, another computative paradox occurs:

although for each contaminated data array a family of analytical solutions exists, the only single solution of the estimation problems is found in modern regression analysis theory.

b) puts the mentioned above extremal values of ξ in (5), one will be able to determine the exact limit of the variation for the linear model parameters a_0 and a_1: $a_0 = 67.77 \pm 0.35$ and $a_1 = 0.865 \pm 0.008$;

c) deletes a set of readings with numbers 1, 7, 8 and 9 from D.I.Mendeleev data array, one will obtain that in the truncated data array $\{y_n, x_n\}^*$ the difference of $|\, y_n - y_n'\, |$ for each n keeps within the limit of the error ε, where ε is the measuring error for readings $\{y_n\}^*$: $\varepsilon = 0.1$. Since in this case $\delta(y - y') \le \varepsilon$, the complete family of analytical solutions has form [6, 7]

$$y'(\xi, x) = g_\alpha(\ 67.566 + 0{,}002\xi + (0.870047 - 0.000097\xi)\, x\ ), \tag{6}$$

where α=0.07; $0 \le \xi \le 45$ and, consequently, $a_0 = 67.521 \pm 0.045$ and $a_1 = 0.872 \pm 0.002$;

d) compares solutions (5) and (6) with the standard LS-solution, one can conclude that LS-estimations of parameter a_0 and a_1 $\{\mathbf{A}' = (67.5{\pm}0.5;\ 0.87{\pm}0.2)\}$ are pretty near equal of the mean values of these parameters in the general analytical solutions (6) and (7). However,

values of variances $\delta a_0'$ and $\delta a_1'$, computed by standard method, disagree with exact values determined by (5).

2.3 The main paradox of regression analysis theory

As it emerges from analysis of information presented in Sect. 2.2, *the main* paradox of modern regression analysis theory is exhibited in a contradiction between this theory statements, which guarantee uniqueness of data analysis problems solution, and multivarious solutions in actual practice. In this section we adduce yet several computative manifestations of this paradox.

Example 2. In table 2 a two-factors simulative data array is presented.

Table 2.

Simulative data array

n	x_n	y_n	n	x_n	y_n	n	x_n	y_n
1	−1.0	0.50	8	−0.3	0.75	15	0.4	0.87
2	−0.9	0.55	9	−0.2	0.77	16	0.5	0.89
3	−0.8	0.59	10	−0.1	0.79	17	0.6	0.90
4	−0.7	0.63	11	0.0	0.81	18	0.7	0.91
5	−0.6	0.66	12	0.1	0.83	19	0.8	0.92
6	−0.5	0.69	13	0.2	0.84	20	0.9	0.93
7	−0.4	0.72	14	0.3	0.86	21	1.0	0.94

Let the approximative model have form

$$y = (a_0 + a_1 x + a_2 x^2)/(1 + a_3 x + a_4 x^2) . \qquad (7)$$

To find vector parameter **A** estimates of the model (7) on the data array from table 2 we use two different estimation methods. As the first method we choose the estimation one, involved in the software CURVE-2.0, designed AISN. In this case we obtain, that

$$\mathbf{A}' = \{0.81;\ 0.008;\ -0.31;\ -0.22;\ -0.24\}.$$

As the second estimation method we select Marquardt method[12]. Using the value **A**′, found above by the first estimation method, as initial value of **A** we obtain that in the second case

$$\mathbf{A}' = \{0.81;\ 0.55;\ 0.035;\ 0.45;\ 0.34\}.$$

Thus,

values of A′, obtained by two different estimation methods, differ from each other.

Example 3. In table 3 yet one two-factors data array is presented. Let us select the model $y = a_1 x + e$ as approximative one and assume, that y is the random variable with the known density function p:

$$p = \exp\{-(y - a_1 x)/(2f(a_2))\}/\sqrt{2\pi f(a_2)}\ , \qquad (8)$$

where $f(a_2) =$ a) a_2; b) $a_2 x^2$; c) $a_2 x$.

We will find estimates of parameters a_1 and a_2 by method of *maximum likelihood*:

$$L = \prod_{n=1}^{N} \exp\{-(y_n - a_1 x_n\} / (2f(a_2)) / \sqrt{2\pi f(a_2)} \quad \Rightarrow \quad \max \tag{9}$$

or $\partial \ln L / \partial a_i = 0$, where symbol "ln" means the natural logarithm.

Table 3.

Two-factors data array

n	x_n	y_n	n	x_n	y_n	n	x_n	y_n
1	5	21	6	11	53	11	16	81
2	6	31	7	11	56	12	19	97
3	8	38	8	12	60	13	20	98
4	8	37	9	14	68	14	22	107
5	10	53	10	15	72	-	-	-

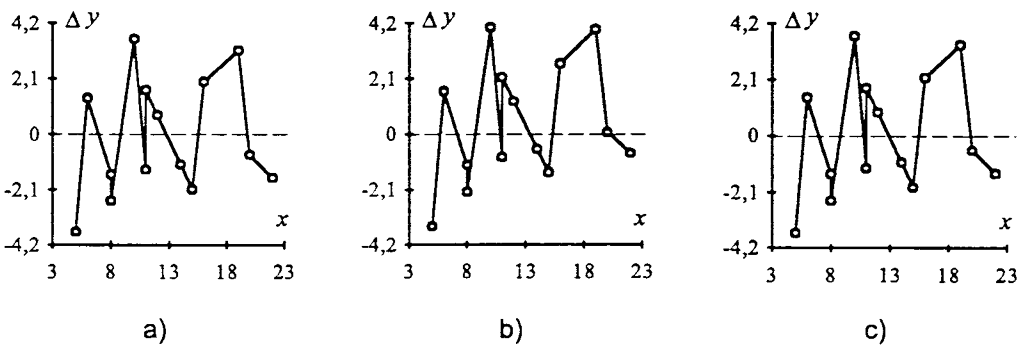


Figure 3. Dependences $\{y_n - a_1'x_n\}$ for different hypothesis about Law for the random variable y variance.

Computation results of V.I.Mudrov and V.P.Kushko[13] show, that in the discussed case the estimates values of parameters a_1 depend on hypothesis about Law for the random variable y variance: for case (a) in (8) $a_1' = 4.938$ ($L' = 4.995 \cdot 10^{-14}$); for case (b) $a_1' = 4.896$ ($L' = 4.421 \cdot 10^{-16}$) and for case (c) $a_1' = 4.927$ ($L' = 9.217 \cdot 10^{-15}$). By analysing obtained results the authors[13] conclude, that, since likelihood function (9) has maximum values for case (a), the more likelihood hypothesis about Law for the random variable y variance is the hypothesis (a): variance of y is the constant value.

We demonstrate in figure 3 that for cases (a), (b) and (c) dependences $\Delta y = \{e_n\} = \{y_n - a_1'x_n\}$ have practically the same form and, consequently,

the strong distinction of values L' for all mentioned cases does not tread on infirm ground.

It should be noted that

– the very apparent expression of the discussed main computative paradox of regression analysis theory one may find also in books[6, 7, 11], where, for the problem on finding the best linear multiple model, fitting Hald data array, a set of solutions,

found by various procedures and statistical tests of modern regression analysis theory, is adduced;

– the most impressive formulation of the main paradox of regression analysis theory is contained in Y.P.Adler introduction [14]:

"When the computation had arisen, the development of regression analysis algorithms went directly «up the stairs, being a descending road». Computer was improving and simultaneously new more advanced algorithms were yielded: whole regression method, step-by-step procedure, stepped method, etc., – it is impossible to name all methods. But again and again it appeared that all these tricks did not allow to obtain a correct solution. At least it became clear that in majority cases the regression problems belonged to a type of incorrect stated problems. Therefore either they can be regularised by exogenous information, or one must put up with ambiguous, multivarious solutions. So the regression analysis degraded ingloriously to the level of a heuristic method, in which the residual analysis and common sense of interpreter play the leading role. Automation of regression analysis problems came to a dead-lock".

3 Data analysis problems at unknown theoretical models

Let us assume, that a researcher is to carry out a quantitative analysis of a data array $\{\mathbf{X}_n\}$ in the absence of theoretical models. Further consideration will be based on the fact [6, 7] that the described situation demands a solution of following problems

– verification of the presence (or absence) of interconnections between analysed properties or phenomena;

– determining (in the case when the interconnection is obvious, a priori and logically plausible) in what force this interconnection is exhibited in comparison with other factors affecting the discussed phenomena;

– drawing a conclusion about the presence of a reliable difference between the selected groups of analysed objects;

– revealing object's characteristics irrelevant to analysed property or phenomenon;

– constructing a regression model describing interconnections between analysed properties or phenomena.

In following sections we consider some methods allowing to solve foregoing problems.

3.1 Correlation analysis

When one is to carry out a quantitative analysis of the data array $\{\mathbf{X}_n\}$ in the absence of theoretical models, it is usual to apply correlation analysis at the earlier

investigation stage, allowing to determine the structure and force of the connections between analysed variables[15-17].

Let, for instance, in an experiment each n-th state of the object be characterised by a pair of its parameters y and x. If relationship between y and x is unknown, it is sometimes possible to establish the existence and nature of their connection by means of such simple way as graphical. Indeed, for realising this way, it is sufficient to construct a plot of the dependence $\{y_n, x_n\}$ in rectangular coordinates $y - x$. In this case the plotted points determine a certain *correlation field*, demonstrating dependences $x = x(y)$ and/or $y = y(x)$ in a visual form.

To characterise the connection between y and x quantitatively one may use *the correlation coefficient* R, determined by the equation[15-17]

$$R_{yx} = \frac{\sum_{n=1}^{N}(y_n - \bar{y})(x_n - \bar{x})}{\sqrt{\sum_{n=1}^{N}(y_n - \bar{y})^2 \sum_{n=1}^{N}(x_n - \bar{x})^2}}. \tag{10}$$

where $\bar{y}$ and $\bar{x}$ are the mean values of parameters y and x computed on all N readings of the array $\{y_n, x_n\}$. It can be demonstrated that absolute value of R_{yx} does not exceed a unit: $-1 \le R_{yx} \le 1$.

If variables y and x are connected by a strict linear dependence $y = a_0 + a_1x$, then $R_{yx} = \pm 1$, where sign of R_{yx} is the same as that of the a_1 parameter. This can follow, for instance, from the fact that, using R_{yx}, one can rewrite the equation for the regression line in the following form[15-17]

$$y = \bar{y} + R_{yx}(S_y/S_x)(x - \bar{x}), \tag{11}$$

where S_y and S_x are mean-square deviations of variables y and x respectively.

In a general case, when $-1 < R_{yx} < 1$, points $\{y_n, x_n\}$ will tend to approach the line (11) more closely with increasing of $|R_{yx}|$ value. Thus, correlation coefficient (10) characterises a linear dependence of y and x rather than an arbitrary one. To illustrate this statement we present in table 4 the values $R_{yx} = R_{yx}(\alpha)$ for the functional dependence $y = x^{\alpha}$, determined on x-interval [0.5; 5.5] in 11 points uniformly.

Table 4.

The values $R_{yx} = R_{yx}(\alpha)$ for the functional dependence $y=x^{\alpha}$, determined on interval [0.5; 5.5]

a	$R_{yx}(\alpha = -a)$	$R_{yx}(\alpha = a)$	a	$R_{yx}(\alpha = -a)$	$R_{yx}(\alpha = a)$
3.0	−0.570	0.927	1.0	−0.795	1.000
2.5	−0.603	0.951	0.5	−0.880	0.989
2.0	−0.650	0.974	0.0	0.0	0.0
1.5	−0.715	0.992	-	-	-

Let us clear up a question what influence has the presence of outliers in the data array $\{y_n, x_n\}$ on the value of correlation coefficient (10). To perform it let us analyse a data array

$$\{x_n\} = (-4; -3; -2; -1; 0; 10), \quad (12)$$

$$\{y_n\} = (2.48; 0.73; -0.04; -1.44; -1.32; 0),$$

where, on simulation conditions[8], the reading with number 6 is *an extremal* outlier (such reading that contrasts sharply from others); approximative function $F(\mathbf{A}, \mathbf{X}) = a_0 + a_1 x$ and $\mathbf{A}_{\text{true}} = (-2; -1)$.

By computing the values of R_{yx} of (10) and s of (1), we determine the number i of a reading, which elimination from this data array leads to the maximum absolute value of R_{yx} and, consequently, to the minimum value of s (the most simple combinatoric-parametric *procedure Ps*, allowing to find one outlier[6, 7] in a data array). Our calculations show that the desirable value $R_{yx} = -0.979$ and $i = 1$. We note, if extremal outlier y_6 is removed from the array (12), $R_{yx} = -0.960$, but s of (1) takes the minimum value. Presented results enable us to state that

procedure Ps loses its effectiveness when revealing the outlier is made not by test s, but by test R_{yx}.

Let us consider another case. For the array (12) the noise array $\{e_n\} = \{-2 - x_n - y_n\} = (-0.48;\ 0.27;\ 0.04;\ 0.44;\ -0.68;\ -12.0)$. We reduce by half the first 5 magnitudes of the noise array $\{e_n\}$: $\{e_n\}_{\text{new}} = (-0.24; 0.14; 0.02; 0.22; -0.34; -12.0)$; form a new array $\{y_n\}_{\text{new}} = \{-2 - x_n - (e_n)_{\text{new}}\}$ and determine again the number i of a reading, which elimination from the data array $\{y_n, x_n\}_{\text{new}}$ leads to the maximum absolute value of R_{yx}. In the described case R_{yx} reaches its maximum absolute value when the reading 6 (extremal outlier) is deleted from the array $\{y_n, x_n\}_{\text{new}}$ ($R_{yx} = -0.989$). If from the array $\{y_n, x_n\}_{\text{new}}$ we eliminate the reading 6, identified correctly by the test "the maximum absolute value of R_{yx}", then by this test we are able to identify correctly the sequent outlier (the reading 5) in the discussed array. Thus, we obtain finally

when a dependence between the analysed variables is to a certain extent close to a linear, one may use the correlation coefficient (10) for revealing outliers, presented in data arrays.

It is known[15 - 17], when the number of analysed variables $K > 2$, the structure and force of the connections between variables $x_1, x_2, \dots, x_K$ are determined by computing all possible pairs of correlation coefficients $R_{x_i x_j}$ from (10). In this case all coefficients $R_{x_i x_j}$ are usually presented in the form of a square symmetric K by K matrix:

$$\mathbf{R} = \begin{bmatrix} R_{11}\ R_{12}\ \dots\ R_{1K} \\ \dots\ \dots\ \dots\ \dots \\ R_{K1}\ R_{K2}\ \dots\ R_{KK} \end{bmatrix}, \quad (13)$$

which is called *a correlation matrix* (we note that in this matrix diagonal elements $R_{ii}= 1$). Finding strong-interconnected pairs of variables $x_1, x_2, \dots, x_K$ on the magnitudes of coefficients R_{ij} from the matrix $\mathbf{R}$ is a traditional use of matrix (13) in data analysis. But, obviously,

using the mentioned way, one should bear in mind all ideas presented above in outline concerning the correlation coefficient (10).

3.2 Discriminant analysis

Let a certain object W be characterised by a value of its vector parameter $\mathbf{X}_w = (x_1, x_2, \dots, x_K)$; $W_1, W_2, \dots, W_p$ be p classes and the object W must be ranged in a class W_j on the value of its vector parameter $\mathbf{X}_w$. In discriminant analysis the formulated problem is the *main* one [18-21].

The accepted technique for solving the mentioned problem entails construction of *a discriminant function* $D(\mathbf{A}, \mathbf{X})$. A form and coefficients $\{a_i\}_{i=1,2,\dots,p}$ values of this function are determined from the requirement, that values of $D(\mathbf{A}, \mathbf{X})$ must have maximum dissimilarity, if parameters of objects, belonging to different populations $W_1, W_2, \dots, W_p$, are used as arguments of this function.

It seems obvious that in a general case, firstly, $D(\mathbf{A}, \mathbf{X})$ may be either linear or non-linear function on $\{a_i\}$ and, secondly, must be some connection between the problem-solving techniques of discriminant and regression analyses. In particular, as stated [18-21], for solving problems of discriminant analysis one may use standard algorithms and programs of regression analysis. Thus, the similarity of techniques, used for solving problems of the regression and discriminant analyses, makes it possible in discriminant analysis to apply alternative algorithms and procedures of regression analysis and, consequently,

if data analysis problems are solved by discriminant analysis techniques then in practice the researcher may meet the same difficulties which are discussed in Sect. 2.

3.3 Regression analysis

In the absence of theoretical models it is usual to employ regression analysis in order to express in a mathematical form the connections existing between variables under analysis.

It happens with extreme frequency that researchers impose limitations on a type and form of approximative models or, in other words, approximative models are often chosen from a given set of ones. Evidently, in this case it is required to solve problem on finding the best approximative model from a given set of models. Let, for instance, it is required to find the best approximative multinomial with a minimal degree. With this in mind, in two examples below we consider some accepted

techniques, used for solving the mentioned problem in approximation and/or regression analysis theories.

Example 4. Let

$$\{x_n\} = \{-1+0.2(n-1)\};\ f(x) = \sin x;\ \{y_n\} = \{\,[\,kf(x_n)\,]\,/\,k\,\}, \tag{14}$$

where square brackets mean the integer part; $n = 1, 2, \ldots, 11$; $f(x)$ is a given function, used for generating the array $\{y_n\}$; a factor $k = 10^3$ and its presence in (14) is necessary for "measuring" all values of y_n within error $\varepsilon = 10^{-3}$. It is required for the presented dependence $\{y_n, x_n\}$ to find the best approximative multinomial with a minimal degree.

A. As well known in approximation theory[22], if the type of the function $f(x)$ is given and either $(m+1)$-derivative of this function weakly varies on the realisation $\{x_n\}$, or on the x-interval $[-1, 1]$ the function $f(x)$ is presented in form of even-converging power series, then the problem of finding the best *even-approximating* multinomial for the discrete dependence $\{y_n, x_n\}$ offers no difficulty. Indeed, in the first case the solution of problem is an interpolative multinomial $P_M(\mathbf{B}, x)$ with a set of Chebyshev points (this multinomial is close to the best even-approximating one). In the second, case one may obtain the solution by the following *economical procedure* of an even-converging power series:

1. Choose the initial part of truncated Taylor series, approximating the function $f(x)$ within error $\varepsilon_M < \varepsilon$, as the multinomial $P_M(\mathbf{B}, x)$ (the multinomial with the degree M and vector parameter $\mathbf{B}$);
2. Replace ε_M with $\varepsilon_M - |b_M|/2^{M-1}$, where b_M is a coefficient of the multinomial $P_M(\mathbf{B}, x)$ at x^M;
3. If $\varepsilon_M > 0$, then replace the multinomial $P_M(\mathbf{B}, x)$ with the multinomial

$$P_{M-1}(x) = P_M(x) - (b_M / 2^{M-1})T_M(x), \tag{15}$$

where $T_M(x)$ is Chebyshev multinomial: $T_0 = 1$, $T_1 = x$ and when $M \geq 2$ $T_M = 2xT_{M-1} - T_{M-2}$. Then decrement M by one and go to point 2. If $\varepsilon_M \leq 0$, then go to point 4;
4. End of computations: the multinomial $P_M(\mathbf{B}, x)$ is the desirable one.

By means of the foregoing economical procedure one may easy obtain that the multinomial with a minimal degree, even-approximating the function $\sin x$, given within error $\varepsilon = 10^{-3}$ on the x-interval $[-1, 1]$, has the following form

$$P_3(x) = (383/384)\,x - (5/32)\,x^3. \tag{16}$$

B. Let $1 \leq M \leq 9$ and in the multinomial $P_\lambda(\mathbf{B}, x) = \sum_{m=0}^{M} \lambda_m b_m x^m$ all $\lambda_m = 1$. By determining LS-estimates of vector parameter $\mathbf{B}$ for each value of M on the formed above array $\{y_n, x_n\}$, we find that in all obtained approximative multinomials $P_\lambda(\mathbf{B}', x)$, as well as in Taylor series of function $\sin x$, the values of coefficients

$b'_{2l} = 0$ at $l = 0, 1, \ldots, 4$. Thus, regression analysis of the discussed array $\{y_n, x_n\}$ allows to determine a form of the best approximative multinomial:

$$P_{2l+1\,\mathrm{opt}}(\mathbf{B}, x) = b_1 x + b_3 x^3 + \ldots + b_{2l+1} x^{2l+1} + \ldots . \qquad (17)$$

We note, if $l = 1$, then the values of parameters b_1 and b_3, computed by regression analysis method (LS-method) for model (17), coincide with ones, shown in (16).

We remind, that in classical variant of regression analysis theory the best approximative multinomial is chosen by the minimal value of the test $s_\lambda = S_\lambda/(N-I_\lambda)$, where S_λ is residual sum-of-squares, $I_\lambda = \Sigma_{m=1}^{M} \lambda_m$; N is total number of readings; λ_m is such characteristic number that $\lambda_m = 0$, if the approximative multinomial $P_\lambda(\mathbf{B}, x)$ does not contain term $b_m x^m$, and $\lambda_m = 1$, otherwise. For approximative models (17) and $l = 0, 1, 2, 3$ the computation values of s are following

l	0	1	2	3
s	0.033	0.00063	0.00043	0.00054

Since s has the minimal value at $l = 2$, for the discussed array in the frame of classical variant of regression analysis theory, the multinomial

$$P_5(x) = b_1 x + b_3 x^3 + b_5 x^5 \qquad (18)$$

is the best approximative one.

C. Since, for each n in the data array (14), the difference of $|y_n - y_n'|$ must be kept within the limit of the error $\varepsilon = 10^{-3}$, the general solution of the discussed problem has the following form[6, 7]

$$y'(\xi, x) = g_\alpha\{(1.0012 - 0.0001\xi)\, x - (0.161200 - 0.000127\xi) x^3\}, \qquad (19)$$

where $\alpha = 0.001$; $0 \le \xi \le 49$ and, consequently, $b_1 = 0.9987 \pm 0.0025$ and $b_2 = -0.1582 \pm 0.0030$.

By analysing solutions (16), (18) and (19) we conclude that in the considered case

the solution of the problem on finding the best fitting multinomial depends on the type of the used mathematical theory.

Example 5. In some software products {for instance, in the different versions of software CURVE, designed by AISN} the solutions of problems on finding the best approximative models are found by the magnitude of *a determination coefficient R*, which value may be computed by a set of formulae

$$R_1 = \sqrt{1 - Q_r / Q},\ Q_r = \Sigma_{n=1}^{N} (y_n - y_n')^2,\ Q = \Sigma_{n=1}^{N} (y_n - \bar{y})^2 \qquad (20)$$

where $\bar{y} = \sum_{n=1}^{N} y_n / N$, y_n is n-th reading of dependent variable, y'_n is n-th value of dependent variable, computed on the fitting model; or by formula (firstly offered by K.Pearson)

$$R_2 = \frac{\sum_{n=1}^{N}(y_n - \bar{y})(y'_n - \bar{y}')}{\sqrt{\sum_{n=1}^{N}(y_n - \bar{y})^2 \sum_{n=1}^{N}(y_n - \bar{y}')^2}} \tag{21}$$

{evidently, one may easy obtain formula (21) from formula (10)}.

Table 5.

The simulative data array to example 5

n	y_n	x_n	n	y_n	x_n
1	85	11	7	205	3.8136396
2	105	5.6002132	8	225	3.5774037
3	125	5.0984022	9	245	3.4193292
4	145	4.7047836	10	265	3.2903451
5	165	4.3936608	11	285	3.1026802
6	185	4.0998636	-	-	-

There is a mathematical proof [23] of the equivalence of formulae (20) and (21). But, if the value of coefficient R_2^2 is computed within error $\geq 10^{-8}$,

in actual practice, for some data arrays, firstly, $R_2^2 \neq R_1^2$ and, secondly, $R_2^2 > 1$.

For instance, if one fits the simulative data array, presented in table 5, by the multinomial $P_M(\mathbf{B}, x)$ with $M = 8$, then software CURVE-2.0 will give the value $R_2^{\,2} = 1.00040$.

4 Problems of quantitative processing experimental dependences found for heterogeneous objects

As it follows from the general consideration[24, 25], in practice at analysis of experimental dependences found for heterogeneous objects, three various situations can be realised: the heterogeneity of investigated objects causes a) no effect; b) a removable (local) inadequacy of postulated fitting model; c) an irremovable (global) inadequacy of the postulated model. In this section we discuss some computative difficulties which may occur at analysis of the mentioned experimental dependences.

Example 6. As we know from Sect. 2.1 if $F(\mathbf{A}, x)$ is a linear model $\{F(\mathbf{A}, x) = \sum_{l=1}^{L} a_l h_l(x)\}$ then the value of $\mathbf{A}'$, minimising residual sum-of-squares S, is

computed by (1). Let rank $\mathbf{H} < L$, or, in other words, there is a linear dependence between columns of matrix $\mathbf{H}$:

$$c_1 h_1 + c_2 h_2 + \ldots + c_L h_L = 0, \tag{22}$$

where at least one coefficient $c_l \neq 0$. In this case matrix $(\mathbf{H}^T\mathbf{H})^{-1}$ does not exist, that means one cannot find $\mathbf{A}'$ from (1). Such situation is known as *strict multicollinearity.*

In the natural science investigations values of the independent variable $\mathbf{X}$ are always determined with a certain round-off error, although this error may be very small. Therefore, if even strict multicollinearity is present, in practice the equation (22) is satisfied only approximately and therefore rank $\mathbf{H} = L$. In such situation application of equation (1) to find the estimate of vector parameter $\mathbf{A}$ gives $\mathbf{A}'$ values drastically deviating from true coefficients values[6, 7, 23].

To correct this situation in regression on *characteristics roots*[26] it is suggested to obtain the information about the grade of matrix $\mathbf{H}^T\mathbf{H}$ conditioning from values of its eigennumbers λ_j and first elements V_{0j} of its eigenvector $\mathbf{V}_j$ and to exclude from regression such j-components, whose eigennumbers λ_j and elements V_{0j} are small. Following values are recommended to use as critical ones: $\lambda_{cr} = 0.05$ and $V_{cr} = 0.1$.

Let us demonstrate, that

in some practical computations the difference between A'_{CHR} and A'_{LS} of (1) can be explained not by the effects of multicollinearity, but by regression model inadequacy, which disappears simultaneously with the effects of multicollinearity after removing outliers.

Indeed, let data array be following

$$\{y_n, \mathbf{X}_n\} = \{1 + 0.5\,n + 0.05\,n^2 + 0.005\,n^3;\ n, n^2, n^3\}, \tag{23}$$

$n = 1, 2, \ldots, 11$ and we introduce two outliers in (23), by means of increasing values y_3 and y_8 on 0.5. For this data array we obtain the following computation results:

$N_{step} = 1, N = 11$: $\mathbf{A}'_{LS} = (1.017; 0.542; 0.0462; 0.0050)$,

$n_a = \{3, 8\}$ $\mathbf{A}'_{CHR} = (1.046; 0.516; 0.0516; 0.0047)$;

$N_{step} = 2, N = 10$: $\mathbf{A}'_{LS} = (0.764; 0.801; -0.0169; 0.0090)$,

$n_a = \{3\}$ $\mathbf{A}'_{CHR} = (0.764; 0.801; -0.0169; 0.0090)$,

where N_{step} is a number of the step in used computative procedure; n_a is a vector to indicate the numbers of anomalous readings, contained in analysing array on the first and second steps of used computative procedure; N is the general quantity of analysing readings. In particular, after the first step of computative procedure from (23) the reading with number 8 is removed; after the second step — readings with numbers 3 and 8. And after the second step the values of $\mathbf{A}'$ are restored without any distortion by both examined algorithms.

By analysing obtained computation results one can conclude that

the difference between A'_{LS} *and* A'_{CHR} *may be caused not only by multicollinearity but also, for instance, by a set of outliers presented in the data array.*

Example 7. In table 6 we adduce an experimental data array, obtained by N.P.Bobrysheva[27], when she investigated magnetic susceptibility (χ, relative units) of polycrystalline system $V_xAl_{1-x}O_{1,5}$ ($x = 0.078$) on the temperature (T, K). Let us consider some computation results of quantitative processing this temperature dependence.

Table 6.

The experimental dependence of magnetic susceptibility of system $V_xAl_{1-x}O_{1,5}$ (x = 0.078) on temperature

n	T_n	χ_n	n	T_n	χ_n	n	T_n	χ_n
1	80	10.97	9	292	3.79	17	501	2.48
2	121	8.06	10	351	3.31	18	512	2.46
3	144	6.94	11	360	3.20	19	523	2.42
4	182	5.56	12	385	3.06	20	559	2.28
5	202	5.11	13	401	3.00	21	601	2.12
6	214	4.75	14	438	2.78	22	651	2.02
7	220	4.62	15	464	2.67	23	668	1.97
8	267	4.00	16	486	2.56	-	-	-

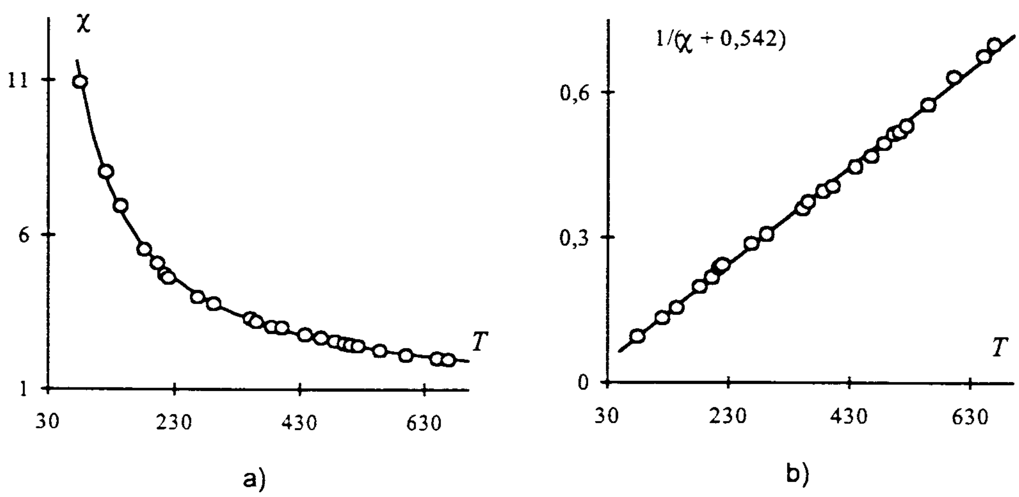


Figure 4. Experimental (circles) and analytical (continuous curves) plots of dependences $\chi(T)$ (a) and $1/(\chi + \chi_2) - T$ (b) for system $V_xAl_{1-x}O_{1,5}$ ($x = 0.078$).

A. In figure 4(*a*, *b*) for the discussed system experimental (circles) and analytical (continuous curves) plots of dependences $\chi(T)$ and $1/(\chi + \chi_2) - T$ are shown. For construction of analytical (continuous) curves we use modified Curie – Weiss law[6, 7, 24, 25]

$$\chi = \chi_0 + C/(T + \theta), \qquad (24)$$

where χ is the experimental magnitude of specific magnetic susceptibility; T is absolute temperature, K; C, θ and χ_0 are parameters: $C = 988$; $\theta = 14$, K; $\chi_0 = 0.54$. From analysing graphical information, presented in figure 4(a, b), one can conclude, that

the magnetic behaviour of system $V_xAl_{1-x}O_{1.5}$ *(x = 0.078) is well explained by the modified Curie – Weiss law (24).*

B. In figure 5(a) the dependence $\Delta\chi = \chi - C/(T + \theta) - \chi_0$ on T for system $V_xAl_{1-x}O_{1,5}$ (x = 0.078) is shown. Since $\Delta\chi_{max} \cong 0.2 >> \varepsilon = 0.01$, where ε is the measurement error in the diccussed experiment, we obtain

in contradiction with the statement of point (A) in this case modified Curie – Weiss law (24) is an inadequate approximative model or, in other words, there is a set of outliers in the analysing experimental dependence.

C. After deleting first 5 readings from the initial data array the parameters values of modified Curie – Weiss law (24) have magnitudes $C = 1386$; $\theta = 89$, K; $\chi_0 = 0.14$. The plot of dependence $\Delta\chi = \chi - C/(T + \theta) - \chi_0$ with foregoing parameters values is shown in figure 5(b).

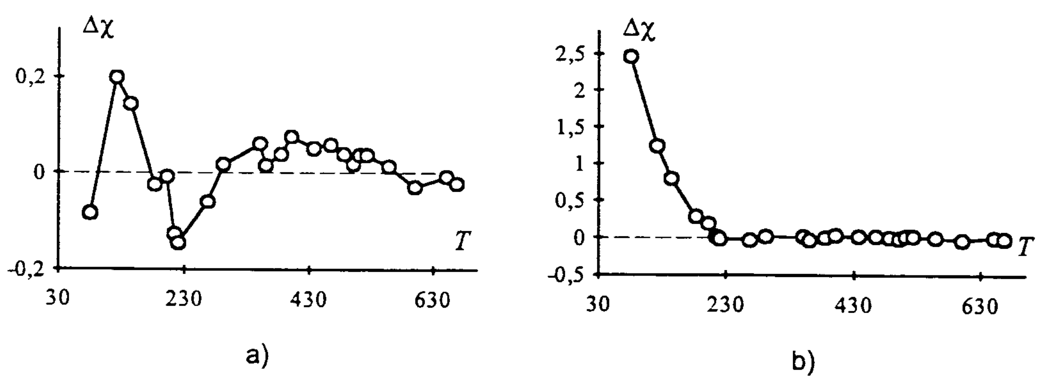


Figure 5. Plots of $\Delta\chi = \chi - C/(T + \theta) - \chi_0$ on T for system $V_xAl_{1-x}O_{1.5}$ (x = 0.078).

Analysing plots $\Delta\chi(T)$, presented in figure 5(a, b), and comparing with each other the parameters values of equation (24), mentioned in points (A) and (C), we conclude, that

neglect of the local inadequacy of the approximative model in the discussed experiment leads to distortion of both form of function $\Delta\chi(T)$ *and parameters values of the modified Curie – Weiss law (24).*

Thus, if, for proving well-fitted properties of equation (24), researchers[27–30] suggest to look at the graphic representation of dependences $\chi(T)$ or $1/\chi(T)$, for the proof completeness one should ask these researchers to present information about the measurement error of values χ and plots $\Delta\chi(T) = \chi - C/(T+\theta) - \chi_0$.

For the sake of convenience, the main causes, given rise to computative difficulties at analysis of experimental dependences found for heterogeneous objects, and methods of their overcoming are adduced in table 7 together. In this table all methods, overcoming computative difficulties, are marked by the symbol Θ, if at present they are *in the rough* or absent in modern data analysis theory.

Table 7.

Main causes and overcoming methods of computative difficulties in modern data analysis theory

	Main causes	Methods of overcoming
1.	Impossibility to take heed of preset measurement accuracy of dependent variable values in the frame of accepted data analysis model	Θ Modification of data analysis model
2.	Limited accuracy of computations	Increasing computation accuracy
3.	Point estimation of parameters	Replacing the point estimation of parameters by interval one
4.	The deficient measurement accuracy of dependent variable values	Increasing measurement accuracy of dependent variable values
5.	Ill-conditioning of estimation problem	Θ Using alternative estimations methods; Increasing measurement accuracy of dependent variable values; Θ Revealing and removing outliers; Designing experiments
6.	Presence of outliers in analysing data arrays	Θ Revealing and removing outliers; Θ Robust estimation of parameters
7.	Inadequacy of approximative model	Θ Eliminating inadequacy of approximative model; Θ Using advanced estimations methods
8.	Finding only single solution of the estimation problems for contaminated data array in the frame of modern data analysis theories	Θ Finding a family of solutions

Using information presented in table 7, let us clear up a question, whether one is able in the frame of modern data analysis theory to obtain reliable solutions for the problems of quantitative processing of experimental dependences, found for heterogeneous objects.

Let, when an investigated object is homogeneous, a connection between characteristics y and $\mathbf{X}$ exist and it be close to functional one: $y = F(\mathbf{A}, \mathbf{X})$. As we said already in beginning of this section, in the discussed experiments three various situations can be realised: the structural heterogeneity

1) has no effect on the experimental dependence $\{y_n, \mathbf{X}_n\}$ or, in other words, in this case it is impossible to distinguish the homogeneous objects from heterogeneous ones on the dependence $\{y_n, \mathbf{X}_n\}$;

2) leads to a distortion of the dependence $\{y_n, \mathbf{X}_n\}$ in some small region $\{\mathbf{X}_{n_1}\} \subset \{\mathbf{X}_n\}$ {the approximative model $F(\mathbf{A}, \mathbf{X})$ has *removable* (local) inadequacy}. In this case for extracting effects, connected with the presence of a homogeneity in the investigated objects, one may use the following way[6, 7]

i) solve the problem on revealing outliers $\{y_{n_1}, \mathbf{X}_{n_1}\}$;

ii) determine the value $\mathbf{A}'$ on readings $\{y_n, \mathbf{X}_n\} \setminus \{y_{n_1}, \mathbf{X}_{n_1}\}$ {we remind, that a set of $\{y_n, \mathbf{X}_n\} \setminus \{y_{n_1}, \mathbf{X}_{n_1}\}$ is to be well-fitted by the model $F(\mathbf{A}, \mathbf{X})$};

iii) detect a type and degree of the effects, connected with the presence of a homogeneity in the investigated objects, on the data array $\{y_{n_1} - F(\mathbf{A}', \mathbf{X}_{n_1}),\ \mathbf{X}_{n_1}\}$.

It follows from point 6 of table 7, that at solving problem (*i*) in actual practice some difficulties, which are unsurmountable in the frame of modern regression analysis theory, can be arisen;

3) leads to a distortion of the dependence $\{y_n, \mathbf{X}_n\}$ in a big region $\{\mathbf{X}_{n_1}\} \subseteq \{\mathbf{X}_n\}$: {the approximative model $F(\mathbf{A}, \mathbf{X})$ has *irremovable* (global) inadequacy}.

It follows from point 7 of table 7, that in this case it is impossible to find a reliable solution of the discussed problem in the frame of modern data analysis theory.

Summarising mentioned in points (1) – (3), we conclude

since at present the methods, marked by the symbol Θ in table 7, are not effective for overcoming computative difficulties or absent in modern data analysis theory, one is not able to obtain reliable solutions for the problems of quantitative processing of experimental dependences found for heterogeneous objects.

From our point of view, one of possible ways, overcoming computative difficulties in modern data analysis theory, is further development of this theory by means of translation of this theory from Aristotelian "binary logic" into more progressive "fuzzy logic"[6, 7, 24, 25, 31, 32].

References

1. Dumitrescu C and Seleacu V *Some notions and questions in number theory* (Erhus University Press, Vail, 1995)
2. Smarandache F *Paradoxist Mathematics* (Pennsylvania, 1985)
3. Linnik Y V *Least squares method and foundations of mathematical-statistical theory of observations processing* (Moscow, Publ. H. of phys.-math. lit., 1958) (In Russian)
4. Rao C P *Linear statistical inference and its applications.* 2nd ed. (New York, Wiley, 1973)
5. Ermakov S M and Zhiglavskiy A A *Mathematical theory of optimal experiment* (Moscow, Nauka, 1987) (In Russian)
6. Chebrakov Y V *The parameters estimation theory in the measuring experiments* (S.–Petersburg, S.–Petersburg State Univ. Press, 1997) (In Russian)

7. Chebrakov Y V and Shmagin V V *Regression data analysis for physicists and chemists* (S.–Petersburg, S.–Petersburg State Univ. Press, 1998)
8. Huber P J *Robust Statistics* (New York, Wiley, 1981)
9. Rousseeuw P J and Leroy A M *Robust regression and outlier detection* (New York, Wiley, 1987)
10. Andrews D F *Technometrics* (1974) **16** 523
11. Draper K R and Smith H *Applied regression analysis* (New York, Wiley, 1981)
12. Gill P E, Murray W and Wright M H *Practical Optimization* (New York – London, Academic Press, 1981)
13. Mudrov V I and Kushko V P *Methods of processing measurements: Quasi- believable estimates* (Moscow, Radio i svyaz, 1983) (In Russian)
14. Adler Y P *The introduction* to Russian edition of the book: Mosteller F and Tukey J W *Data analysis and regression* (Moscow, Finansy i statistika, 1982) (In Russian)
15. Aivazyan S A, Yenyukov I S and Meshalkin L D *Applied statistics. Analysis of dependences* (Moscow, Finansy i statistika, 1985) (In Russian)
16. Bendat J S and Piersol A G *Random data: Analysis and measurement procedures* (New York, Wiley, 1986)
17. Rozanov Y A *Probability theory, random processes and mathematical statistics: A textbook for higher education.* (Moscow, Nauka, 1989) (In Russian)
18. Afifi A A and Azen S P *Stastistical analysis: A computer-oriented approach.* 2nd ed. (New York – London, Academic Press, 1979)
19. Lloyd E and Lederman W *Handbook of applicable mathematics.* Volume IV: *Statistics.* Parts A and B (New York, Wiley, 1984)
20. Mandel I D *Cluster analysis* (Moscow, Nauka, 1988) (In Russian)
21. Seber G A F *Multivariate observations* (New York, Wiley, 1984)
22. Laurent P J *Approximation et optimisation* (Paris, Hermann, 1972)
23. Vuchkov I, Boyadzieva L and Solakov E *Applied linear regression analysis* (Moscow, Finansy i statistika, 1981) (In Russian)
24. Chebrakov Y V and Shmagin V V *J. Izv. Vysh. Uch. Zav. Fizika* (1995) **5** 40; (1998) **6** 110 (Engl. Transl. *Russian Phys. J.* (1995) **5** 470; (1998) **6**)
25. Chebrakov Y V *J. Izv. Vysh. Uch. Zav. Fizika* (1997) **7** 103 (Engl. Transl. *Russian Phys. J.* (1997) **7** 687)
26. Webster J, Gunst R and Mason R *Technometrics* (1974) **16** 513
27. Bobrysheva N P *Magnetic dilution of a series of oxides containing elements of the first transitional period*: Condensed version of the PhD thesis (Leningrad, 1974) (In Russian)
28. Amine K, Tressand A, Imoto H et al. *J. of Solid State Chemistry* (1992) **96** 287
29. Miyata N, Kimishima Y, Akutsu N and Oguro I *J. Magnetism and Magnetic Materials* (1990) **90&91** 337
30. Zayachuk D M, Ivanchuk R D, Kempnik V I and Mikitjuk V I *Fizika tverdogo tela* (1996) **36** № 8 2502 (In Russian)
31. Chebrakov Y V and Shmagin V V *Correct solutions of fit problems in different experimental situations* In: Advanced Mathematical Tools in Metrology III (Singapore, World Scientific, 1997) 246
32. Chebrakov Y V and Shmagin V V *Some nontraditional approaches to solving a few data analysis problems* In: Integral methods in science and engineering (Edinburg, Longman, 1997) **2** 69.

THE NORMAL BEHAVIOR OF THE SMARANDACHE FUNCTION

Kevin Ford

Let $S(n)$ be the smallest integer k so that $n|k!$. This is known as the Smarandache function and has been studied by many authors. If $P(n)$ denotes the largest prime factor of n, it is clear that $S(n) \geqslant P(n)$. In fact, $S(n) = P(n)$ for most n, as noted by Erdös [E]. This means that the number, $N(x)$, of $n \leqslant x$ for which $S(n) \neq P(n)$ is $o(x)$. In this note we prove an asymptotic formula for $N(x)$.

First, denote by $\rho(u)$ the Dickman function, defined by

$$\rho(u) = 1 \quad (0 \leqslant u \leqslant 1), \qquad \rho(u) = 1 - \int_1^u \frac{\rho(v-1)}{v}\,dv \quad (u > 1).$$

For $u > 1$ let $\xi = \xi(u)$ be defined by

$$u = \frac{e^{\xi} - 1}{\xi}.$$

It can be easily shown that

$$\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right),$$

where $\log_k x$ denotes the kth iterate of the logarithm function. Finally, let $u_0 = u_0(x)$ be defined by the equation

$$\log x = u_0^2 \xi(u_0).$$

The function $u_0(x)$ may also be defined directly by

$$\log x = u_0 \left(x^{1/u_0^2} - 1\right).$$

It is straightforward to show that

$$(1) \qquad u_0 = \left(\frac{2\log x}{\log_2 x}\right)^{\frac{1}{2}} \left(1 - \frac{\log_3 x}{2\log_2 x} + \frac{\log 2}{2\log_2 x} + O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^2\right)\right).$$

We can now state our main result.

Theorem 1. *We have*

$$N(x) \sim \frac{\sqrt{\pi}(1+\log 2)}{2^{3/4}}(\log x \log_2 x)^{3/4} x^{1\ \ 1/u_0}\rho(u_0).$$

There is no way to write the asymptotic formula in terms of "simple" functions, but we can get a rough approximation.

Corollary 2. *We have*

$$N(x) = x \exp\left\{-(\sqrt{2}+o(1))\sqrt{\log x \log_2 x}\right\}.$$

The asymptotic formula can be made a bit simpler, without reference to the function ρ as follows.

Corollary 3. *We have*

$$N(x) \sim \frac{e^{\gamma}(1+\log 2)}{2\sqrt{2}}(\log x)^{\frac{1}{2}}(\log_2 x) x^{1\ \ 2/u_0} \exp\left\{\int_0^{\frac{\log x}{u_0^2}} \frac{e^v-1}{v}\,dv\right\},$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.

This will follow from Theorem 1 using the formula in Lemma 2 which relates $\rho(u)$ and $\xi(u)$.

The distribution of $S(n)$ is very closely related to the distribution of the function $P(n)$. We begin with some standard estimates of the function $\Psi(x,y)$, which denotes the number of integers $n \leqslant x$ with $P(n) \leqslant y$.

Lemma 1 [HT, Theorem 1.1]. *For every $\epsilon > 0$,*

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right), \quad u = \frac{\log x}{\log y},$$

uniformly in $1 \leqslant u \leqslant \exp\{(\log y)^{3/5\ \ \epsilon}\}$.

Lemma 2 [HT, Theorem 2.1]. *For $u \geqslant 1$,*

$$\begin{aligned}\rho(u) &= \left(1+O\left(\frac{1}{u}\right)\right)\sqrt{\frac{\xi'(u)}{2\pi}}\exp\left\{\gamma - \int_1^u \xi(t)\,dt\right\}\\ &= \exp\left\{-u\left(\log u + \log_2 u - 1 + O\left(\frac{\log_2 u}{\log u}\right)\right)\right\}.\end{aligned}$$

Lemma 3 [HT, Corollary 2.4]. *If $u > 2$, $|v| \leqslant u/2$, then*

$$\rho(u-v) = \rho(u)\exp\{v\xi(u) + O((1+v^2)/u)\}.$$

Further, if $u > 1$ and $0 \leqslant v \leqslant u$ then

$$\rho(u-v) \ll \rho(u)e^{v\xi(u)}.$$

We will show that most of the numbers counted in $N(x)$ have

$$P(n) \approx \exp\left\{\sqrt{\tfrac{1}{2}\log x \log_2 x}\right\}.$$

Let

$$Y_1 = \exp\left\{\tfrac{1}{3}\sqrt{\log x \log_2 x}\right\}, \quad Y_2 = Y_1^6 = \exp\left\{2\sqrt{\log x \log_2}\right\}.$$

Let N_1 be the number of n counted by $N(x)$ with $P(n) \leqslant Y_1$, let N_2 be the number of n with $P(n) \geqslant Y_2$, and let $N_3 = N(x) - N_1 - N_2$. By Lemmas 1 and 2,

$$N_1 \leqslant \Psi(x, Y_1) = x\exp\{-(1.5 + o(1))\sqrt{\log x \log_2 x}\}.$$

For the remaining $n \leqslant x$ counted by $N(x)$, let $p = P(n)$. Then either $p^2|n$ or for some prime $q < p$ and $b \geqslant 2$ we have $q^b \parallel n$, $q^b \nmid p!$. Since $p!$ is divisible by $q^{[p/q]}$ and $b \leqslant 2\log x$, it follows that $q > p/(3\log x) > p^{1/2}$. In all cases n is divisible by the square of a prime $\geqslant Y_2/(3\log x)$ and therefore

$$N_2 \leqslant \sum_{p \geqslant \frac{Y_2}{3\log x}} \frac{x}{p^2} \leqslant \frac{6x\log x}{Y_2} \ll x\exp\left\{-1.9\sqrt{\log x \log_2 x}\right\}.$$

Since $q > p^{1/2}$ it follows that $q^{[p/q]} \parallel p!$. If n is counted by N_3, there is a number $b \geqslant 2$ and prime $q \in [p/b, p]$ so that $q^b|n$. For each $b \geqslant 2$, let $N_{3,b}(x)$ be the number of n counted in N_3 such that $q^b \parallel n$ for some prime $q \geqslant p/b$. We have

$$\sum_{b \geqslant 6} N_{3,b} \ll x\left(\frac{3\log x}{Y_1}\right)^5 \ll x\exp\left\{-(5/3 + o(1))\sqrt{\log x \log_2 x}\right\}.$$

Next, using Lemma 1 and the fact that ρ is decreasing, for $3 \leqslant b \leqslant 5$ we have

$$\begin{aligned}
N_{3,b} &= \sum_{Y_1 < p < Y_2}\left(\Psi\left(\frac{x}{p^b}, p\right) + \sum_{p/b \leqslant q < p}\Psi\left(\frac{x}{pq^b}, q\right)\right) \\
&\ll x\sum_{Y_1 < p < Y_2}\left(\frac{1}{p^b}\rho\left(\frac{\log x}{\log p} - b\right) + \sum_{p/2 < q < p}\frac{1}{pq^b}\rho\left(\frac{\log x - \log p - b\log q}{\log p}\right)\right) \\
&\ll x\sum_{Y_1 < p < Y_2} p^{\quad b}\rho\left(\frac{\log x}{\log p} - (b+1)\right).
\end{aligned}$$

By partial summation, the Prime Number Theorem, Lemma 2 and some algebra,

$$N_{3,b} \ll \exp\left\{-(1.5 + o(1))\sqrt{\log x \log_2 x}\right\}.$$

The bulk of the contribution to $N(x)$ will come from $N_{3,2}$. Using Lemma 1 we obtain
(2)

$$N_{3,2} = \sum_{Y_1<p<Y_2} \left(\Psi\left(\frac{x}{p^2}, p\right) + \sum_{\frac{p}{2}<q<p} \Psi\left(\frac{x}{pq^2}, q\right)\right)$$

$$= \left(1+O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right) x \sum_{Y_1<p<Y_2} \left(\frac{\rho\left(\frac{\log x}{\log p}-2\right)}{p^2} + \sum_{p/2<q<p} \frac{\rho\left(\frac{\log x \;\, \log p}{\log q}-2\right)}{pq^2}\right).$$

By Lemma 3, we can write

$$\rho\left(\frac{\log x - \log p}{\log q}-2\right) = \rho\left(\frac{\log x}{\log q}-3\right)\left(1+O\left(\sqrt{\frac{\log_2 x}{\log x}}\right)\right).$$

The contribution in (2) from p near Y_1 or Y_2 is negligible by previous analysis, and for fixed $q \in [Y_1, Y_2/2]$ the Prime Number Theorem implies

$$\sum_{q<p<2q} \frac{1}{p} = \frac{\log 2}{\log q} + O((\log q)^{\ 2}) = \frac{\log 2}{\log p} + O\left(\frac{1}{\log^2 Y_1}\right).$$

Reversing the roles of p, q in the second sum in (2), we obtain

$$N_{3,2} = \left(1+O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right) x \sum_{Y_1<p<Y_2} \frac{1}{p^2}\left(\rho\left(\frac{\log x}{\log p}-2\right) + \frac{\log 2}{\log p}\rho\left(\frac{\log x}{\log p}-3\right)\right).$$

By partial summation, the Prime Number Theorem with error term, and the change of variable $u = \log x/\log p$,

$$(3) \qquad N_{3,2} = \left(1+O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right) x \int_{u_1}^{u_2} \left(\frac{\rho(u-2)}{u} + \frac{\log 2}{\log x}\rho(u-3)\right) x^{\ 1/u} du,$$

where

$$u_1 = \frac{1}{2}\sqrt{\frac{\log x}{\log_2 x}}, \qquad u_2 = 6u_1.$$

The integrand attains its maximum value near $u = u_0$ and we next show that the most of the contribution of the integral comes from u close to u_0. Let

$$w_0 = \frac{u_0}{100}, \quad w_1 = K\sqrt{u_0}, \quad w_2 = w_1\left(\frac{\log_3 x}{\log_2 x}\right)^{1/2},$$

where K is a large absolute constant. Let I_1 be the contribution to the integral in (3) with $|u-u_0| > w_0$, let I_2 be the contribution from $w_1 < |u-u_0| \leqslant w_0$, let I_3 be the contribution from $w_2 < |u-u_0| \leqslant w_1$, and let I_4 be the contribution from $|u-u_0| \leqslant w_2$. First, by Lemma 2, the integrand in (3) is

$$\exp\left\{-\left(\frac{1}{c}-\frac{c}{2}+o(1)\right)\sqrt{\log x \log_2 x}\right\}, \quad c = \left(\frac{\log_2 x}{\log x}\right) u.$$

The function $1/c + c/2$ has a minimum of $\sqrt{2}$ at $c = \sqrt{2}$, so it follows that

$$I_1 \ll \exp\left\{-\left(\sqrt{2} + 10^{\ 5}\right)\sqrt{\log x \log_2 x}\right\}.$$

Let $u = u_0 - v$. For $w_1 \leqslant |v| \leqslant w_0$, Lemma 2 and the definition (1) of u_0 imply that the integrand in (3) is

$$\begin{aligned}
&\leqslant \rho(u_0)\exp\left\{v\xi(u_0) - \frac{\log x}{u_0}\left(1 + \frac{v}{u_0} + \frac{v^2}{u_0^2} + \frac{v^3}{u_0^3}\right) + O\left(\frac{v^2}{u_0} + \log u_0\right)\right\} \\
&\ll \rho(u_0)x^{\ 1/u_0}\exp\left\{-\frac{v^2}{u_0^3}\log x + O\left(\frac{v^2}{u_0} + \log u_0\right)\right\} \\
&\ll \rho(u_0)x^{\ 1/u_0}\exp\left\{-0.9\frac{v^2}{u_0^3}\log x\right\}
\end{aligned}$$

for K large enough. It follows that

$$I_2 \ll u_0\rho(u_0)x^{\ 1/u_0}\exp\{-20\log_2 x\} \ll (\log x)^{\ 10}\rho(u_0)x^{\ 1/u_0}.$$

For the remaining u, we first apply Lemma 3 with $v = 2$ and $v = 3$ to obtain

$$I_3 + I_4 = \left(1 + O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right)\int_{u_0\ \ w_1}^{u_0+w_1} \rho(u)x^{\ 1/u}\left(\frac{e^{2\xi(u)}}{u} + \frac{\log 2}{\log x}e^{3\xi(u)}\right)du$$

We will show that $I_3 + I_4 \gg \rho(u_0)x^{\ 1/u_0}(\log x)^{3/2}$, which implies

$$N(x) = \left(1 + O\left(\sqrt{\tfrac{\log_2 x}{\log x}}\right)\right)\int_{u_0\ \ w_1}^{u_0+w_1} \rho(u)x^{\ 1/u}\left(\frac{e^{2\xi(u)}}{u} + \frac{\log 2}{\log x}e^{3\xi(u)}\right)du. \tag{4}$$

This provides an asymptotic formula for $N(x)$, but we can simplify the expression somewhat at the expense of weakening the error term. First, we use the formula

$$\xi(u) = \log u + \log_2 u + O\left(\frac{\log_2 u}{\log u}\right),$$

and then use $u = u_0 + O(u_0^{1/2})$ and (1) to obtain

$$I_3 + I_4 = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right)\frac{\sqrt{2}}{4}(1 + \log 2)x(\log x)^{\frac{1}{2}}(\log_2 x)^{\frac{3}{2}}\int_{u_0\ \ w_1}^{u_0+w_1} \rho(u)x^{\ 1/u}\,du.$$

By Lemma 3, when $w_2 \leqslant |v| \leqslant w_1$, where $u = u_0 - v$, we have

$$\begin{aligned}
\rho(u_0 - v)x^{\ \frac{1}{u_0\ \ v}} &\ll \rho(u_0)x^{\ \frac{1}{u_0}}\exp\left\{v\xi(u_0) - \frac{\log x}{u_0}\left(\frac{v}{u_0} + \frac{v^2}{u_0^2} + \frac{v^3}{u_0^3}\right)\right\} \\
&\ll \rho(u_0)x^{\ \frac{1}{u_0}}\exp\left\{-\frac{v^2}{u_0^3}\log x\right\} \\
&\ll \rho(u_0)x^{\ \frac{1}{u_0}}\exp\left\{-\frac{w_2^2}{u_0^3}\log x\right\} \\
&\ll \rho(u_0)x^{\ \frac{1}{u_0}}(\log_2 x)^{\ 3}
\end{aligned}$$

provided K is large enough. This gives

$$\int_{w_2 \leqslant |u-u_0| \leqslant w_1} \rho(u) x^{-1/u} \, du \ll \rho(u_0) x^{-1/u_0} (\log x)^{1/4} (\log_2 x)^{-3.5}.$$

For the remaining v, Lemma 3 gives

$$\rho(u_0 - v) x^{-1/(u_0-v)} = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right) \rho(u_0) x^{-1/u_0} \exp\left\{-\frac{v^2}{u_0^3} \log x\right\}.$$

Therefore,

$$\rho(u_0)^{-1} x^{\frac{1}{u_0}} \int_{u_0-w_2}^{u_0+w_2} \rho(u) x^{-1/u} \, du = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right) \int_{-w_2}^{w_2} \exp\left\{-v^2 \frac{\log x}{u_0^3}\right\} dv.$$

The extension of the limits of integration to $(-\infty, \infty)$ introduces another factor $1 + O((\log_2 x)^{-1})$, so we obtain

$$I_3 + I_4 = \left(1 + O\left(\tfrac{\log_3 x}{\log_2 x}\right)\right) \frac{\sqrt{\pi}(1 + \log 2)}{2^{3/4}} (\log x \log_2 x)^{3/4} \rho(u_0) x^{-\frac{1}{u_0}}$$

and Theorem 1 follows. Corollary 2 follows immediately from Theorem 1 and (1). To obtain Corollary 3, we first observe that $\xi'(u) \sim u^{-1}$ and next use Lemma 2 to write

$$\rho(u_0) \sim \frac{e^\gamma}{\sqrt{2\pi u_0}} \exp\left\{-\int_1^{u_0} \xi(t) \, dt\right\}.$$

By the definitions of ξ and u_0 we then obtain

$$\begin{aligned}
\int_1^{u_0} \xi(t) \, dt &= \int_0^{\xi(u_0)} e^v - \frac{e^v - 1}{v} \, dv \\
&= e^{\xi(u_0)} - 1 - \int_0^{\xi(u_0)} \frac{e^v - 1}{v} \, dv \\
&= \frac{\log x}{u_0} - \int_0^{\frac{\log x}{u_0^2}} \frac{e^v - 1}{v} \, dv.
\end{aligned}$$

Corollary 3 now follows from (1).

References

[E] P. Erdős, *Problem 6674*, Amer. Math. Monthly **98** (1991), 965.

[HT] A. Hildebrandt and G. Tenenbaum, *Integers without large prime factors*, J. Théor. Nombres Bordeaux **5** (1993), 411–484.

On values of arithmetical functions at factorials I

J. Sándor

Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

1. The Smarandache function is a characterization of factorials, since $S(k!) = k$, and is connected to values of other arithmetical functions at factorials. Indeed, the equation

$$S(x) = k \quad (k \geq 1 \text{ given}) \tag{1}$$

has $d(k!) - d((k-1)!)$ solutions, where $d(n)$ denotes the number of divisors of n. This follows from $\{x : \ S(x) = k\} = \{x : \ x|k!, \ x \nmid (k-1)!\}$. Thus, equation (1) always has at least a solution, if $d(k!) > d((k-1)!)$ for $k \geq 2$. In what follows, we shall prove this inequality, and in fact we will consider the arithmetical functions $\varphi, \sigma, d, \omega, \Omega$ at factorials. Here $\varphi(n)$ = Euler's arithmetical function, $\sigma(n)$ = sum of divisors of n, $\omega(n)$ = number of distinct prime factors of n, $\Omega(n)$ = number of total divisors of n. As it is well known, we have $\varphi(1) = d(1) = 1$, while $\omega(1) = \Omega(1) = 0$, and for $1 < \prod_{i=1}^{r} p_i^{a_i}$ ($a_i \geq 1$, p_i distinct primes) one has

$$\varphi(n) = n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right),$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1},$$

$$\omega(n) = r,$$

$$\Omega(n) = \sum_{i=1}^{r} a_i,$$

$$d(n) = \prod_{i=1}^{r} (a_i + 1). \tag{2}$$

The functions φ, σ, d are multiplicative, ω is additive, while Ω is totally additive, i.e. φ, σ, d satisfy the functional equation $f(mn) = f(m)f(n)$ for $(m,n) = 1$, while ω, Ω satisfy the equation $g(mn) = g(m) + g(n)$ for $(m,n) = 1$ in case of ω, and for all m, n is case of Ω (see [1]).

2. Let $m = \prod_{i=1}^{r} p_i^{\alpha_i}$, $n = \prod_{i=1}^{r} p_i^{\beta_i}$ $(\alpha_i, \beta_i \geq 0)$ be the canonical factorizations of m and n. (Here some α_i or β_i can take the values 0, too). Then

$$d(mn) = \prod_{i=1}^{r} (\alpha_i + \beta_i + 1) \geq \prod_{i=1}^{r} (\beta_i + 1)$$

with equality only if $\alpha_i = 0$ for all i. Thus:

$$d(mn) \geq d(n) \tag{3}$$

for all m, n, with equality only for $m = 1$.

Since $\prod_{i=1}^{r} (\alpha_i + \beta_i + 1) \leq \prod_{i=1}^{r} (\alpha_i + 1) \prod_{i=1}^{r} (\beta_i + 1)$, we get the relation

$$d(mn) \leq d(m)d(n) \tag{4}$$

with equality only for $(n,m) = 1$.

Let now $m = k$, $n = (k-1)!$ for $k \geq 2$. Then relation (3) gives

$$d(k!) > d((k-1)!) \text{ for all } k \geq 2, \tag{5}$$

thus proving the assertion that equation (1) always has at least a solution (for $k = 1$ one can take $x = 1$).

With the same substitutions, relation (4) yields

$$d(k!) \leq d((k-1)!)d(k) \text{ for } k \geq 2 \tag{6}$$

Let $k = p$ (prime) in (6). Since $((p-1)!, p) = 1$, we have equality in (6):

$$\frac{d(p!)}{d((p-1)!)} = 2, \quad p \text{ prime}. \tag{7}$$

3. Since $S(k!)/k! \to 0$, $\dfrac{S(k!)}{S((k-1)!)} = \dfrac{k}{k-1} \to 1$ as $k \to \infty$, one may ask the similar problems for such limits for other arithmetical functions.

It is well known that

$$\frac{\sigma(n!)}{n!} \to \infty \text{ as } n \to \infty. \tag{8}$$

In fact, this follows from $\sigma(k) = \sum_{d|k} d = \sum_{d|k} \dfrac{k}{d}$, so

$$\frac{\sigma(n!)}{n!} = \sum_{d|n!} \frac{1}{d} \geq 1 + \frac{1}{2} + \ldots + \frac{1}{n} > \log n,$$

as it is known.

From the known inequality ([1]) $\varphi(n)\sigma(n) \leq n^2$ it follows

$$\frac{n}{\varphi(n)} \geq \frac{\sigma(n)}{n},$$

so $\dfrac{n!}{\varphi(n!)} \to \infty$, implying

$$\frac{\varphi(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{9}$$

Since $\varphi(n) > d(n)$ for $n > 30$ (see [2]), we have $\varphi(n!) > d(n!)$ for $n! > 30$ (i.e. $n \geq 5$), so, by (9)

$$\frac{d(n!)}{n!} \to 0 \text{ as } n \to \infty. \tag{10}$$

In fact, much stronger relation is true, since $\dfrac{d(n)}{n^\varepsilon} \to 0$ for each $\varepsilon > 0$ $(n \to \infty)$ (see [1]). From $\dfrac{d(n!)}{n!} < \dfrac{\varphi(n!)}{n!}$ and the above remark on $\sigma(n!) > n! \log n$, it follows that

$$\limsup_{n\to\infty} \frac{d(n!)}{n!} \log n \leq 1. \tag{11}$$

These relations are obtained by very elementary arguments. From the inequality $\varphi(n)(\omega(n)+1) \geq n$ (see [2]) we get

$$\omega(n!) \to \infty \text{ as } n \to \infty \tag{12}$$

and, since $\Omega(s) \geq \omega(s)$, we have

$$\Omega(n!) \to \infty \text{ as } n \to \infty. \tag{13}$$

From the inequality $nd(n) \geq \varphi(n) + \sigma(n)$ (see [2]), and (8), (9) we have

$$d(n!) \to \infty \text{ as } n \to \infty. \tag{14}$$

This follows also from the known inequality $\varphi(n)d(n) \geq n$ and (9), by replacing n with $n!$. From $\sigma(mn) \geq m\sigma(n)$ (see [3]) with $n = (k-1)!$, $m = k$ we get

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \geq k \quad (k \geq 2) \tag{15}$$

and, since $\sigma(mn) \leq \sigma(m)\sigma(n)$, by the same argument

$$\frac{\sigma(k!)}{\sigma((k-1)!)} \leq \sigma(k) \quad (k \geq 2). \tag{16}$$

Clearly, relation (15) implies

$$\lim_{k\to\infty} \frac{\sigma(k!)}{\sigma((k-1)!)} = +\infty. \tag{17}$$

From $\varphi(m)\varphi(n) \leq \varphi(mn) \leq m\varphi(n)$, we get, by the above remarks, that

$$\varphi(k) \leq \frac{\varphi(k!)}{\varphi((k-1)!)} \leq k, \quad (k \geq 2) \tag{18}$$

implying, by $\varphi(k) \to \infty$ as $k \to \infty$ (e.g. from $\varphi(k) > \sqrt{k}$ for $k > 6$) that

$$\lim_{k\to\infty} \frac{\varphi(k!)}{\varphi((k-1)!} = +\infty. \tag{19}$$

By writing $\sigma(k!) - \sigma((k-1)!) = \sigma((k-1)!)\left[\frac{\sigma(k!)}{\sigma((k-1)!)} - 1\right]$, from (17) and $\sigma((k-1)!) \to \infty$ as $k \to \infty$, we trivially have:

$$\lim_{k\to\infty} [\sigma(k!) - \sigma((k-1)!)] = +\infty. \tag{20}$$

In completely analogous way, we can write:

$$\lim_{k\to\infty} [\varphi(k!) - \varphi((k-1)!)] = +\infty. \tag{21}$$

4. Let us remark that for $k = p$ (prime), clearly $((k-1)!, k) = 1$, while for $k =$ composite, all prime factors of k are also prime factors of $(k-1)!$. Thus

$$\omega(k!) = \begin{cases} \omega((k-1)!k) = \omega((k-1)!) + \omega(k) & \text{if } k \text{ is prime} \\ \omega((k-1)!) & \text{if } k \text{ is composite } (k \geq 2). \end{cases}$$

Thus

$$\omega(k!) - \omega((k-1)!) = \begin{cases} 1, & \text{for } k = \text{prime} \\ 0, & \text{for } k = \text{composite} \end{cases} \tag{22}$$

Thus we have

$$\begin{aligned} &\limsup_{k\to\infty} [\omega(k!) - \omega((k-1)!)] = 1 \\ &\liminf_{k\to\infty} [\omega(k!) - \omega((k-1)!)] = 0 \end{aligned} \tag{23}$$

Let p_n be the nth prime number. From (22) we get

$$\frac{\omega(k!)}{\omega((k-1)!)} - 1 = \begin{cases} \frac{1}{n-1}, \text{if } k = p_n \\ 0, \text{if } k = \text{composite}. \end{cases}$$

Thus, we get

$$\lim_{k\to\infty} \frac{\omega(k!)}{\omega((k-1)!)} = 1. \tag{24}$$

The function Ω is totally additive, so

$$\Omega(k!) = \Omega((k-1)!k) = \Omega((k-1)!) + \Omega(k),$$

giving

$$\Omega(k!) - \Omega((k-1)!) = \Omega(k). \tag{25}$$

This implies

$$\limsup_{k\to\infty}[\Omega(k!) - \Omega((k-1)!)] = +\infty \tag{26}$$

(take e.g. $k = 2^m$ and let $m \to \infty$), and

$$\liminf_{k\to\infty}[\Omega(k!) - \Omega((k-1)!)] = 2$$

(take $k =$ prime).

For $\Omega(k!)/\Omega((k-1)!)$ we must evaluate

$$\frac{\Omega(k)}{\Omega((k-1)!)} = \frac{\Omega(k)}{\Omega(1) + \Omega(2) + \ldots + \Omega(k-1)}.$$

Since $\Omega(k) \le \dfrac{\log k}{\log 2}$ and by the theorem of Hardy and Ramanujan (see [1]) we have

$$\sum_{n\le x} \Omega(n) \sim x \log\log x \quad (x \to \infty)$$

so, since $\dfrac{\log k}{(k-1)\log\log(k-1)} \to 0$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \frac{\Omega(k!)}{\Omega((k-1)!)} = 1. \tag{27}$$

5. Inequality (18) applied for $k = p$ (prime) implies

$$\lim_{p\to\infty} \frac{1}{p} \cdot \frac{\varphi(p!)}{\varphi((p-1)!)} = 1. \tag{28}$$

This follows by $\varphi(p) = p - 1$. On the other hand, let $k > 4$ be composite. Then, it is known (see [1]) that $k|(k-1)!$. So $\varphi(k!) = \varphi((k-1)!k) = k\varphi((k-1)!)$, since $\varphi(mn) = m\varphi(n)$ if $m|n$. In view of (28), we can write

$$\lim_{k\to\infty} \frac{1}{k} \cdot \frac{\varphi(k!)}{\varphi((k-1)!)} = 1. \tag{29}$$

For the function σ, by (15) and (16), we have for $k = p$ (prime) that $p \leq \dfrac{\sigma(p!)}{\sigma((p-1)!)} \leq \sigma(p) = p+1$, yielding

$$\lim_{p\to\infty} \frac{1}{p} \cdot \frac{\sigma(p!)}{\sigma((p-1)!)} = 1. \tag{30}$$

In fact, in view of (15) this implies that

$$\liminf_{k\to\infty} \frac{1}{k} \cdot \frac{\sigma(k!)}{\sigma((k-1)!)} = 1. \tag{31}$$

By (6) and (7) we easily obtain

$$\limsup_{k\to\infty} \frac{d(k!)}{d(k)d((k-1)!)} = 1. \tag{32}$$

In fact, inequality (6) can be improved, if we remark that for $k = p$ (prime) we have $d(k!) = d((k-1)!) \cdot 2$, while for $k =$ composite, $k > 4$, it is known that $k|(k-1)!$. We apply the following

Lemma. *If $n|m$, then*

$$\frac{d(mn)}{d(m)} \leq \frac{d(n^2)}{d(n)}. \tag{33}$$

Proof. Let $m = \prod p^{\alpha} \prod q^{\beta}$, $n = \prod p^{\alpha'}$ $(\alpha' \leq \alpha)$ be the prime factorizations of m and n, where $n|m$. Then

$$\frac{d(mn)}{d(m)} = \frac{\prod(\alpha+\alpha'+1)\prod(\beta+1)}{\prod(\alpha+1)\prod(\beta+1)} = \prod\left(\frac{\alpha+\alpha'+1}{\alpha+1}\right).$$

Now $\dfrac{\alpha+\alpha'+1}{\alpha+1} \leq \dfrac{2\alpha'+1}{\alpha'+1} \Leftrightarrow \alpha' \leq \alpha$ as an easy calculations verifies. This immediately implies relation (33).

By selecting now $n = k$, $m = (k-1)!$, $k > 4$ composite we can deduce from (33):

$$\frac{d(k!)}{d((k-1)!)} \leq \frac{d(k^2)}{d(k)}. \tag{34}$$

By (4) we can write $d(k^2) < (d(k))^2$, so (34) represents indeed, a refinement of relation (6).

References

[1] T.M. Apostol, *An introduction to analytic number theory*, Springer Verlag, 1976.

[2] J. Sándor, *Some diophantine equations for particular arithmetic functions* (Romanian), Univ. Timişoara, Seminarul de teoria structurilor, No.53, 1989, pp.1-10.

[3] J. Sándor, *On the composition of some arithmetic functions*, Studia Univ. Babeş-Bolyai Math. **34**(1989), 7-14.

THE AVERAGE VALUE OF THE SMARANDACHE FUNCTION

Steven R. Finch
MathSoft Inc.
101 Main Street
Cambridge, MA, USA 02142
sfinch@mathsoft.com

Given a positive integer n, let $P(n)$ denote the largest prime factor of n and $S(n)$ denote the smallest integer m such that n divides $m!$

The function $S(n)$ is known as the Smarandache function and has been intensively studied [1]. Its behavior is quite erratic [2] and thus all we can reasonably hope for is a statistical approximation of its growth, e.g., an average. It appears that the sample mean $E(S)$ satisfies [3]

$$E(S(N)) = \frac{1}{N} \cdot \sum_{n=1}^{N} S(n) = O\left(\frac{N}{\ln(N)}\right)$$

as N approaches infinity, but I don't know of a rigorous proof. A natural question is if some other sense of average might be more amenable to analysis.

Erdös [4,5] pointed out that $P(n) = S(n)$ for almost all n, meaning

$$\lim_{N\to\infty} \frac{\left|\{n \le N : P(n) < S(n)\}\right|}{N} = 0 \qquad \text{that is,} \qquad \left|\{n \le N : P(n) < S(n)\}\right| = o(N)$$

as N approaches infinity. Kastanas [5] proved this to be true, hence the following argument is valid. On one hand,

$$\lambda = \lim_{n\to\infty} E\left(\frac{\ln(P(n))}{\ln(n)}\right) \le \lim_{n\to\infty} E\left(\frac{\ln(S(n))}{\ln(n)}\right) = \lim_{N\to\infty} \frac{1}{N} \cdot \sum_{n=1}^{N} \frac{\ln(S(n))}{\ln(n)}$$

The above summation, on the other hand, breaks into two parts:

$$\lim_{N\to\infty} \frac{1}{N} \cdot \left(\sum_{P(n)=S(n)} \frac{\ln(P(n))}{\ln(n)} + \sum_{P(n)<S(n)} \frac{\ln(S(n))}{\ln(n)} \right)$$

The second part vanishes:

$$\lim_{N\to\infty}\frac{1}{N}\cdot\left(\sum_{P(n)<S(n)}\frac{\ln(S(n))}{\ln(n)}\right)\leq \lim_{N\to\infty}\frac{1}{N}\cdot\left(\sum_{P(n)<S(n)}1\right)=\lim_{N\to\infty}\frac{o(N)}{N}=0$$

while the first part is bounded from above:

$$\lim_{N\to\infty}\frac{1}{N}\cdot\left(\sum_{P(n)=S(n)}\frac{\ln(P(n))}{\ln(n)}\right)\leq \lim_{N\to\infty}\frac{1}{N}\cdot\sum_{n=1}^{N}\frac{\ln(P(n))}{\ln(n)}=\lim_{n\to\infty}E\left(\frac{\ln(P(n))}{\ln(n)}\right)=\lambda$$

We deduce that

$$\lim_{n\to\infty}E\left(\frac{\ln(S(n))}{\ln(n)}\right)=\lambda=0.6243299885\ldots$$

where λ is the famous Golomb-Dickman constant [6-9]. Therefore $\lambda \cdot n$ is the asymptotic average number of digits in the output of S at an n-digit input, that is, 62.43% of the original number of digits. As far as I know, this result about the Smarandache function has not been published before.

A closely related unsolved problem concerns estimating the variance of S.

References

1. C. Dumitrescu and V. Seleacu, *The Smarandache Function*, Erhus University Press, 1996.
2. N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, website URL http://www.research.att.com/~njas/sequences/, AT&T Research (sequences A002034 and A006530).
3. S. Tabirca and T. Tabrica, Some upper bounds of Smarandache's function, *Smarandache Notions Journal* 8 (1997) 205-211.
4. P. Erdös, personal communication to T. Yau (1995), in *Smarandache Notions Journal* 8 (1997) 220-224.
5. I. Kastanas, The smallest factorial that is a multiple of *n*, *Amer. Math. Monthly* 101 (1994) 179.
6. L. A. Shepp and S. P. Lloyd, Ordered cycle lengths in a random permutation, *Trans. Amer. Math. Soc.* 121 (1966) 350-557.
7. D. E. Knuth and L. Trabb Pardo, Analysis of a simple factorization algorithm, *Theoret. Comp. Sci.* 3 (1976) 321-348.
8. D. E. Knuth, *The Art of Computer Programming*, vol. 2, *Seminumerical Algorithms* 2nd ed., Addison Wesley 1981 (pp. 367-368, 395 and 611).
9. S. R. Finch, *Favorite Mathematical Constants*, website URL http://www.mathsoft.com/asolve/constant/constant.html, MathSoft Inc.

On the Irrationality of Certain Constants Related to the Smarandache Function

J. Sándor

Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

1. Let $S(n)$ be the Smarandache function. Recently I. Cojocaru and S. Cojocaru [2] have proved the irrationality of $\sum_{n=1}^{n} \frac{S(n)}{n!}$.

The author of this note [5] showed that this is a consequence of an old irrationality criteria (which will be used here once again), and proved a result implying the irrationality of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{S(n)}{n!}$.

E. Burton [1] has studied series of type $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$, which has a value $\in \left(e - \frac{5}{2}, \frac{1}{2}\right)$. He showed that the series $\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!}$ is convergent for all $r \in \mathbf{N}$. I. Cojocaru and S. Cojocaru [3] have introduced the "third constant of Smarandache" namely $\sum_{n=2}^{\infty} \frac{1}{S(2)S(3)\dots S(n)}$, which has a value between $\frac{71}{100}$ and $\frac{97}{100}$. Our aim in the following is to prove that the constants introduced by Burton and Cojocaru-Cojocaru are all irrational.

2. The first result is in fact a refinement of an old irraionality criteria (see [4] p.5):

Theorem 1. *Let (x_n) be a sequence of nonnegative integers having the properties:*

(1) there exists $n_0 \in \mathbf{N}^*$ such that $x_n \le n$ for all $n \ge n_0$;

(2) $x_n < n - 1$ for infinitely many n;

(3) $x_m > 0$ for an infinity of m.

Then the series $\sum_{n=1}^{\infty} \frac{x_n}{n!}$ *is irrational.*

Let now $x_n = S(n-1)$. Then

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!} = \sum_{n=3}^{\infty} \frac{x_n}{n!}.$$

Here $S(n-1) \leq n-1 < n$ for all $n \geq 2$; $S(m-1) < m-2$ for $m > 3$ composite, since by $S(m-1) < \frac{2}{3}(m-1) < m-2$ for $m > 4$ this holds true. (For the inequality $S(k) < \frac{2}{3}k$ for $k > 3$ composite, see [6]). Finally, $S(m-1) > 0$ for all $m \geq 1$. This proves the irrationality of $\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!}$.

Analogously, write

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+r)!} = \sum_{m=r+2}^{\infty} \frac{S(m-r)}{m!}.$$

Put $x_m = S(m-r)$. Here $S(m-r) \leq m-r < m$, $S(m-r) \leq m-r < m-1$ for $r \geq 2$, and $S(m-r) > 0$ for $m \geq r+2$. Thus, the above series is irrational for $r \geq 2$, too.

3. The third constant of Smarandache will be studied with the following irrationality criterion (see [4], p.8):

Theorem 2. *Let $(a_n), (b_n)$ be two sequences of nonnegative integers satisfying the following conditions:*

(1) $a_n > 0$ for an infinity of n;

(2) $b_n \geq 2$, $0 \leq a_n \leq b_n - 1$ for all $n \geq 1$;

(3) there exists an increasing sequence (i_n) of positive integers such that

$$\lim_{n\to\infty} b_{i_n} = +\infty, \quad \lim_{n\to\infty} a_{i_n}/b_{i_n} = 0.$$

Then the series $\sum_{n=1}^{\infty} \frac{a_n}{b_1 b_2 \dots b_n}$ *is irrational.*

Corollary. *For $b_n \geq 2$, (b_n positive integers), (b_n) unbounded the series* $\sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \dots b_n}$ *is irrational.*

Proof. Let $a_n \equiv 1$. Since $\limsup_{n\to\infty} b_n = +\infty$, there exists a sequence (i_n) such that $b_{i_n} \to \infty$. Then $\frac{1}{b_{i_n}} \to 0$, and the three conditions of Theorem 2 are verified.

By selecting $b_n \equiv S(n)$, we have $b_p = S(p) = p \to \infty$ for p a prime, so by the above Corollary, the series $\sum_{n=1}^{\infty} \frac{1}{S(1)S(2)\dots S(n)}$ is irrational.

References

[1] E. Burton, *On some series involving Smarandache function*, Smarandache Function J. **6**(1995), no.1, 13-15.

[2] I. Cojocaru and S. Cojocaru, *The second constant of Smarandache*, Smarandache Notions J. **7**(1996), no.1-2-3, 119-120.

[3] I. Cojocaru and S. Cojocaru, *The third and fourth constants of Smarandache*, Smarandache Notions J. **7**(1996), no.1-2-3, 121-126.

[4] J. Sándor, *Irrational Numbers* (Romanian), Univ. Timişoara, Caiete Metodico-Stiinţifice No.44, 1987, pp. 1-18.

[5] J. Sándor, *On the irrationality of certain alternative Smarandache series*, Smarandache Notion J. **8**(1997), no.1-2-3, 1997, 143-144.

[6] T. Yau, *A problem of maximum*, Smarandache Function J., vol. **4-5**(1994), no.1, p.45.

Smarandache Magic Squares

*Sabin Tabirca**

*Bucks University College, Computing Department

The objective of this article is to investigate the existence of magic squares made with Smarandache's numbers [Tabirca, 1998]. Magic squares have been studied intensively and many aspects concerning them have been found. Many interesting things about magic squares can be found at the following WEB page http://www.pse.che.tohoko.ac.jp/~msuzuki/MagicSquares.html.

***Definition* 1.** A Smarandache magic square is a square matrix $a \in M_n(N)$ with the following properties:

a) $\left\{a_{i,j} \mid i,j=\overline{1,n}\right\} = \left\{S(i) \mid i=\overline{1,n^2}\right\}$ (1)

b) $\left(\forall\, j=\overline{1,n}\right)\sum_{i=1}^{n} a_{i,j} = k$ (2)

c) $\left(\forall\, i=\overline{1,n}\right)\sum_{j=1}^{n} a_{i,j} = k$ (3)

Therefore, a Smarandache magic square is a square matrix by order n that contains only the elements $S(1), S(2), \ldots, S(n^2)$ [Smarandache, 1980] and satisfies the sum properties (2-3). According to these properties, the sum of elements on each row or column should be equal to the same number k. Obviously, this number satisfies the following equation $k = \frac{\sum_{i=1}^{n^2} S(i)}{n}$.

***Theorem* 1.** If the equation $n \mid \sum_{i=1}^{n^2} S(i)$ does not hold, then there is not a Smarandache magic square by order n.

Proof

This proof is obvious by using the simple remark $k = \frac{\sum_{i=1}^{n^2} S(i)}{n} \in N$. If $a \in M_n(N)$ is a Smarandache magic square, then the equation $n \mid \sum_{i=1}^{n^2} S(i)$ should hold. Therefore, if this equation does not hold, there is no a Smarandache magic square. ♣

Theorem 1 provides simple criteria to find the non-existence of Smarandache magic square. All the numbers $1<n<101$ that do not satisfy the equation $n \mid \sum_{i=1}^{n^2} S(i)$ can be found by using a simple computation [Ibstedt, 1997]. They are {2, 3, ..., 100}\{6, 7, 9, 58, 69}. Clearly, a Smarandache magic square does not exist for this numbers. If n is one of the numbers 6, 7, 9, 58, 69 then the equation $n \mid \sum_{i=1}^{n^2} S(i)$ holds [see Table 1]. This does not mean necessarily that there is a Smarandache magic square. In this case, a Smarandache magic square is found using other techniques such us detailed analysis or exhaustive computation.

n	S(n)	$\sum_{i=1}^{n^2} S(i)$
6	3	330
7	7	602
9	6	1413
58	29	1310162
69	23	2506080

Table 1. The values of n that satisfy $n \mid \sum_{i=1}^{n^2} S(i)$.

An algorithm to find a Smarandache magic square is proposed in the following. This algorithm uses *Backtracking* strategy to complete the matrix a that satisfies (1-3). The going trough matrix is done line by line from the left-hand side to the right-hand side.

The algorithm computes:

- Go trough the matrix
  - Find an unused element of the set $S(1), S(2), \ldots, S(n^2)$ for the current cell.
  - If there is no an unused element, then compute a step back.
  - If there is an used element, then
    - Put this element in the current cell.
    - Check the backtracking conditions.
    - If they are verified and the matrix is full, then a Smarandache magic square has been found.
    - If they are verified and the matrix is not full, then compute a step forward.

```
procedure Smar_magic_square(n);
begin
col:=1; row:=1;a[col, row]:=0;
while row>0 do begin
   while a[col, row]<n*n do begin
         a[col,row]:=a[col,row]+1;
         call check(col,row,n,a,cont);
         if cont=0 then exit;
   end
   if cont =0 then call back(col,row);
   if cont=1 and col=n and row=n
            then call write_square(n,a)
             else call forward(col,row);
end;
write('result negative');
end;

procedure back(col, row);
begin
   col:=col-1;
   if col=0 then begin
         col:=n;row:=row-1;
   end;
end;

procedure forward(col, row);
begin
   col:=col+1;
   if col=n+1 then begin
          col:=1;row:=row+1;
   end;
end;

procedure write_square(n,a);
begin
  for i:=1 to n do begin
     for j:=1 to n do write (S(a[i,j]),' ');
     writeln;
  end;
  stop;
end;

procedure check(col,row,n,,k,a,cont);
begin
cont:=1; sum:=0;
for i:=1 to col do sum:=sum+S(a[i,j]);
if (sum>k) or (col=n and sum<>k) then begin
          cont:=0;
```

```
        return;
end;
sum:=0
for j:=1 to row do sum:=sum+S(a[i,j]);
if (sum>k) or (row=n and sum<>k) then begin
                cont:=0;
                return;
        end;
end;
```

Figure 1. Detailed algorithm for Smarandache magic squares.

The backtracking conditions are the following:

$$\left(\forall j=\overline{1,n}\right)\sum_{i=1}^{col} a_{i,j} \le k \text{ and } \left(\forall j=\overline{1,n}\right)\sum_{i=1}^{n} a_{i,j} = k \tag{4a}$$

$$\left(\forall i=\overline{1,n}\right)\sum_{j=1}^{row} a_{i,j} \le k \text{ and } \left(\forall i=\overline{1,n}\right)\sum_{j=1}^{n} a_{i,j}' = k . \tag{4b}$$

$$\left(\forall (i,j)<(row,col)\right) a_{i,j} \ne a_{row,col} \tag{5}$$

These conditions are checked by the procedure check. A detailed algorithm is presented in a pseudo-cod description in Figure 1.

***Theorem* 2**. If there is a Smarandache magic square by order n, then the procedure Smar_magic_square finds it.

Proof

This theorem establishes the correctness property of the procedure Smar_magic_square. The *Backtracking* conditions are computed correctly by the procedure check that verifies if the equations (4-5) hold. The correctness this algorithm is given by the correctness of the *Backtracking* strategy. Therefore, this procedure finds a Smarandache magic square.

♣

***Theorem* 3.** The complexity of the procedure complexity Smar_magic_square is $O(n^{2\cdot n^2+1})$.

Proof

The complexity of the procedure Smar_magic_square is studied in the worst case when there is not a Smarandache magic square. In this case, this procedure computes all the checking operations for the *Backtracking* strategy. Therefore, all the values

$S(1), S(2), \ldots, S(n^2)$ are gone through for each cell. For each value put into a cell, at most *O(n)* operations are computed by the procedure check. Therefore, the complexity is $O\left(n \cdot (n^2)^{n^2}\right) = O\left(n^{2 \cdot n^2 + 1}\right)$. ♣

***Remark* 1.** The complexity $O(n^{2 \cdot n^2 + 1})$ is not polynomial. Moreover, because this is a very big complexity, the procedure Smar_magic_square can be applied only for small values of n. For example, this procedure computes at most $6^{73} > 10^{56}$ operations in the case n=6. The above procedure has been translated into a C program that has been run on a Pentium MMX 233 machine. The execution of this program has taken more than 4 hours for n=6. Unfortunately, there is not a Smarandache magic square for this value of n. The result of computation for n=7 has not been provided by computer after a twelve hours execution. This reflects the huge number of operations that should be computed ($7^{99} > 10^{83}$).

According to these negative results, we believe that Smarandache magic squares do not exist. If n is a big number that satisfy the equation $n \mid \sum_{i=1}^{n^2} S(i)$, then we have many possibilities to change, to permute and to arrange the numbers $S(1), S(2), \ldots, S(n^2)$ into a square matrix. In spite of that Equations (2-3) cannot be satisfied. Therefore, we may conjecture the following: ***"There are not Smarandache magic squares"***. In order to confirm or infirm this conjecture, we need more powerful method than the above computation. Anyway, the computation looks for a particular solution, therefore it does not solve the problem.

References

Ibstedt, H. (1997) *Surfing on the Ocean of Numbers - a few Smarandache Notions and Similar Topics*, Erhus University Press.

Smarandache, F (1980) A Function in Number Theory, *Analele Univ. Timisoara*, **XVIII**.

Tabirca, S (1998) Problem 24 – Smarandache magic squares, Smarandache Notion Journal, **Vol 9 – nr 1-2.**

Some inequalities concerning Smarandache's function

Sabin Tabirca* ***Tatiana Tabirca*****

*Bucks University College, Computing Department

**Transilvania University of Brasov, Computer Science Department

The objectives of this article are to study the sum $\sum_{d|n} S(d)$ and to find some upper bounds for Smarandache's function. This sum is proved to satisfy the inequality $\sum_{d|n} S(d) \le n$ at most all the composite numbers. Using this inequality, some new upper bounds for Smarandache's function are found. These bounds improve the well-known inequality $S(n) \le n$.

1. Introduction

The object that is researched is Smarandache's function. This function was introduced by Smarandache [1980] as follows:

$$S: N^* \to N \text{ defined by } S(n) = \min\{k \in N \mid k! = \underline{M}n\} \quad (\forall n \in N^*). \tag{1}$$

The following main properties are satisfied by S :

$$(\forall a,b \in N^*)\ (a,b) = 1 \Rightarrow S(a \cdot b) = \max\{S(a), S(b)\}. \tag{2}$$

$$(\forall a \in N^*)\ S(a) \le a \text{ and } S(a) = a \text{ iif } a \text{ is prim}. \tag{3}$$

$$(\forall p \in N^*, p \text{ prime})(\forall k \in N^*)\ S(p^k) \le p \cdot k. \tag{4}$$

Smarandache's function has been researched for more than 20 years, and many properties have been found. Inequalities concerning the function S have a central place and many articles have been published [Smarandache, 1980], [Cojocaru, 1997], [Tabirca, 1997], [Tabirca, 1988]. Two important directions can be identified among these inequalities. First direction and the most important is represented by the inequalities concerning directly the function S such as upper and lower bounds. The second direction is given by the inequalities involving sums or products with the function S.

2. About the sum $\sum_{d|n} S(d)$

The aim of this section is to study the sum $\sum_{d|n} S(d)$.

Let $SS(n) = \sum_{d|n} S(d)$ denote the above sum. Obviously, this sum satisfies $SS(n) = \sum_{1 \neq d|n} S(d)$. Table 1 presents the values of $S(n)$ and $SS(n)$ for $n<50$ [Ibstedt, 1997]. From this table, it can be seen that the inequality $SS(n) \le n+2$ holds for all n=1, 2, ..., 50 and $n \neq 12$. Moreover, if n is a prim number, then the inequality becomes equality $SS(n) = n$.

***Remarks* 1.**

a) If n is a prime number, then $SS(n) = S(1) + S(n) = n$.

b) If $n>2$ is a prim number, then

$$SS(2 \cdot n) = S(1) + S(2) + S(n) + S(2 \cdot n) = 2 + n + n = 2 \cdot n + 2,$$

c) $SS(n^2) = S(1) + S(n) + S(n^2) = n + 2 \cdot n = 3 \cdot n \le n^2$.

N	S	SS	n	S	SS	n	S	SS	n	S	SS	n	S	SS
1	0	0	11	11	11	21	7	17	31	31	31	41	41	41
2	2	2	12	4	16	22	11	24	32	8	24	42	7	36
3	3	3	13	13	13	23	23	23	33	11	25	43	43	43
4	4	6	14	7	16	24	4	24	34	17	36	44	11	39
5	5	5	15	5	13	25	10	15	35	7	19	45	6	25
6	3	8	16	6	16	26	13	28	36	6	34	46	23	48
7	7	7	17	17	17	27	9	18	37	37	37	47	47	47
8	4	10	18	6	20	28	7	27	38	19	40	48	6	36
9	6	9	19	19	19	29	29	29	39	13	29	49	14	21
10	5	12	20	5	21	30	5	28	40	5	30	50	10	32

Table 1. The values of n, S, SS.

The inequality $SS(n) \le n$ is proved to be true for the following particular values $n = p^k, 2 \cdot p^k, 3 \cdot p^k$ and $6 \cdot p^k$.

***Lemma* 1.** If $p>2$ is a prime number and $k>1$, then the inequality $SS(p^k) \le p^k$ holds.

Proof

The following inequality holds according to inequality (4) and the definition of SS.

$$SS(p^k) = \sum_{i=1}^{k} S(p^i) \le \sum_{i=1}^{k} p \cdot i = p \cdot \frac{k \cdot (k+1)}{2}.$$

The inequality

$$\sum_{i=1}^{k} p \cdot i = p \cdot \frac{k \cdot (k+1)}{2} \le p^k \tag{5}$$

is proved to be true by analysing the following cases.

- $k=2 \Rightarrow 3 \cdot p \le p^2$. (6)
- $k=3 \Rightarrow 6 \cdot p \le p^3$. (7)
- $k=4 \Rightarrow 10 \cdot p \le p^4$. (8)

Inequalities (6-8) are true because $p>2$.

- $k>4 \Rightarrow p^k \ge p \cdot p^{k-1} \ge p \cdot 2^{k-1} = p \cdot \sum_{i=0}^{k-1} \binom{k-1}{i}$. The first and the last three terms of this sum are kept and it is found $p^k \ge p \cdot \left(2 \cdot \binom{k-1}{0} + 2 \cdot \binom{k-1}{1} + 2 \cdot \binom{k-1}{2} \right) = p \cdot (k^2 - k + 2)$. The inequality $k^2 - k + 2 \ge \frac{k \cdot (k+1)}{2}$ holds because $k>4$, therefore $p^k \ge p \cdot \frac{k \cdot (k+1)}{2}$ is true.

Therefore, the inequality $S(p^k) \le p^k$ holds. ♣

***Remark* 2.** The inequality $S(p^k) \le p^k$ is still true for $p=2$ and $k>3$ because (8) holds for these values. Table 1 shows that the inequality is not true for $p=2$ and $k=2,3$.

***Lemma* 2.** If $p>2$ is a prime number and $k>1$, then the inequality $SS(2 \cdot p^k) \le 2 \cdot p^k$ holds.

Proof

The definition of SS gives the following equation

$$SS(p^k) = S(2) + \sum_{i=1}^{k} S(p^i) + \sum_{i=1}^{k} S(2 \cdot p^i).$$

Applying the inequality $S(2 \cdot p^i) \le p \cdot i$ and (4), we have

$$SS(2 \cdot p^k) \le 2 + \sum_{i=1}^{k} p \cdot i + \sum_{i=1}^{k} p \cdot i = 2 + p \cdot k \cdot (k+1). \tag{9}$$

The inequality

$$2 + p \cdot k \cdot (k+1) \le 2 \cdot p^k \tag{10}$$

is proved to be true as before.

- $k=2 \Rightarrow 2 + 6 \cdot p \le 2 \cdot p^2$. (11)
- $k=3 \Rightarrow 2 + 12 \cdot p \le 2 \cdot p^3$. (12)
- $k=4 \Rightarrow 2 + 20 \cdot p \le 2 \cdot p^4$. (13)
- $k=5 \Rightarrow 2 + 30 \cdot p \le 2 \cdot p^5$. (14)
- $k=6 \Rightarrow 2 + 42 \cdot p \le 2 \cdot p^5$. (15)

These above inequalities (11-15) are true because $p>2$.

- $k>6 \Rightarrow \ p^k \ge p \cdot p^{k-1} \ge p \cdot 2^{k-1} = p \cdot \sum_{i=0}^{k-1} \binom{k-1}{i}$. The first and the last fourth terms

  of this sum are kept finding

$$p^k \ge p \cdot \left(2 \cdot \binom{k-1}{0} + 2 \cdot \binom{k-1}{1} + 2 \cdot \binom{k-1}{2} + 2 \cdot \binom{k-1}{3} \right) \ge$$

$$\ge p \cdot \left(2 \cdot \binom{k-1}{0} + 2 \cdot \binom{k-1}{1} + 2 \cdot \binom{k-1}{2} + 2 \cdot \binom{k-1}{2} \right) =$$

$$= p \cdot \left(2 \cdot k^2 - 4 \cdot k + 4 \right) \ge 2 + p \cdot (k^2 + k)$$

The last inequality holds because $k>6$, therefore $2 \cdot p^k \ge 2 + p \cdot k \cdot (k+1)$ is true.

The inequality $SS(2 \cdot p^k) \le 2 \cdot p^k$ holds because (10) has been found to be true.

♣

Remark 3. Similarly, the inequality $SS(3 \cdot p^k) \le 3 \cdot p^k$ can be proved for all ($p>3$ and $k \ge 1$) or ($p=2$ and $k \ge 3$).

***Lemma* 3.** If $p>3$ is a prime number and $k\geq 1$, then the inequality $SS(6\cdot p^k)\leq 6\cdot p^k$ holds.

Proof

The starting point is given by the following equation (16)

$$SS(6\cdot p^k)=S(2)+S(3)+S(6)+\sum_{i=1}^{k}S(p^i)+\sum_{i=1}^{k}S(2\cdot p^i)+\sum_{i=1}^{k}S(3\cdot p^i)+\sum_{i=1}^{k}S(6\cdot p^i). \tag{16}$$

The inequalities $S(p^i), S(2\cdot p^i), S(3\cdot p^i), S(6\cdot p^i)\leq p\cdot i$ hold for all $i>1$ because $p\geq 5$. Therefore, the inequality

$$SS(6\cdot p^k)\leq 8+\sum_{i=1}^{k}p\cdot i+\sum_{i=1}^{k}p\cdot i+\sum_{i=1}^{k}p\cdot i+\sum_{i=1}^{k}p\cdot i=8+4\cdot\sum_{i=1}^{k}p\cdot i \tag{17}$$

holds. The inequality $SS(6\cdot p^k)\leq 8+4\cdot p^k\leq 6\cdot p^k$ is found to be true by applying (5) in (17).

♣

The following propositions give the main properties of the function *SS*. Let *d(n)* denote the number of divisors of *n*.

***Proposition* 1.** If *a* is natural numbers such that $S(a)\geq 4$, then the inequality $S(a)\geq 2\cdot d(a)$ holds.

Proof

The proof is made directly as follows:

$$S(a)=\sum_{1\neq d|a}S(d)=\sum_{1,n\neq d|a}S(d)+S(a)\geq\sum_{1,n\neq d|a}2+S(a)=2\cdot(d(a)-2)+S(a)=$$

$$=2\cdot d(a)+S(a)-4\geq 2\cdot d(a).$$ ♣

***Remark* 4.** The inequality $S(a)\geq 4$ is verified for all the numbers $a\geq 4$ and $a\neq 6$.

***Proposition* 2.** If *a*, *b* are two natural numbers such that $(a,b)=1$, then the inequality $SS(a\cdot b)\leq d(a)\cdot SS(b)+d(b)\cdot SS(a)$ holds.

Proof

This proof is made by using (2) and the simple remark that $a,b\geq 0\Rightarrow\max\{a,b\}\leq a+b$.

The set of the divisors of ab is split into three sets as follows:

$$\{1\neq d \mid a\cdot b = \underline{M}d\} =$$
$$\{1\neq d \mid a=\underline{M}d\}\cup\{1\neq d \mid b=\underline{M}d\}\cup\{d_1d_2 \mid a=\underline{M}d_1\neq 1\wedge b=\underline{M}d_2\neq 1\wedge(d_1,d_2)=1\}. \qquad (18)$$

The following transformations hold according to (18).

$$SS(a\cdot b)=\sum_{\{1\neq d \mid a\cdot b = \underline{M}d\}} S(d)=\sum_{\{1\neq d \mid a=\underline{M}d\}} S(d)+\sum_{\{1\neq d \mid b=\underline{M}d\}} S(d)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} S(d_1\cdot d_2)=$$

$$=SS(a)+SS(b)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} \max\{S(d_1),S(d_2)\}\leq$$

$$\leq SS(a)+SS(b)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} [S(d_1)+S(d_2)]=$$

$$=SS(a)+SS(b)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} S(d_1)+\sum_{\{1\neq d_1 \mid a=\underline{M}d_1\}}\ \sum_{\{1\neq d_2 \mid b=\underline{M}d_2\}} S(d_2)=$$

$$=SS(a)+SS(b)+SS(a)\cdot[d(b)-1]+SS(b)\cdot[d(a)-1]$$

Therefore, the inequality $SS(a\cdot b)\leq d(a)\cdot SS(b)+d(b)\cdot SS(a)$ holds. ♣

***Proposition* 3.** If a, b are two natural numbers such that $S(a), S(b)\geq 4$ and $(a,b)=1$, then the inequality $SS(a\cdot b)\leq SS(a)\cdot SS(b)$ holds.

Proof

Proposition 1-2 are applied to prove this proposition as follows:

$$S(a), S(b)\geq 4 \Rightarrow S(a)\geq 2\cdot d(a) \text{ and } S(b)\geq 2\cdot d(b) \qquad (19)$$

$$(a,b)=1 \Rightarrow SS(a\cdot b)\leq d(a)\cdot SS(b)+d(b)\cdot SS(a). \qquad (20)$$

The proof is completed if the inequality $d(a)\cdot SS(b)+d(b)\cdot SS(a)\leq SS(a)\cdot SS(b)$ is found to be true. This is given by the following equivalence

$$d(a)\cdot SS(b)+d(b)\cdot SS(a)\leq SS(a)\cdot SS(b)\Leftrightarrow$$

$$d(a)\cdot d(b)\leq [SS(a)-d(a)]\cdot[SS(b)-d(b)].$$

This last inequality holds according to (19).

Therefore, the inequality $SS(a\cdot b)\leq SS(a)\cdot SS(b)$ is true. ♣

Theorem 1. If n is a natural number such that $n\neq 8, 12, 20$ then

a) $SS(n)=n+2$ if $(\exists\ p \text{ prime})\ n=2\cdot p$. (21)

b) $SS(n)\leq n$, otherwise. (22)

Proof

The proof of this theorem is made by using the induction on n.

Equation (21) is true according to Remark 1.a. Table 1 shows that Equation (22) holds for $n<51$ and $n \neq 8, 12, 20$. Let $n>51$ be a natural number. Let us suppose that Equation (9) is true for all the number k that satisfies $k<n$ and k does not have the form $k=2p$, p prime. The following cases are analysed:

- ***n* is prime** $\Rightarrow SS(n)=n$, therefore Equation (9) holds.
- ***n*=2*p*, *p*>2 prime** $\Rightarrow SS(n)=n+2$, therefore Equation (21) holds.
- **($n = 2^k$ and k>3) or ($n = p^k$ and *k*>1)** $\Rightarrow SS(n) \leq n$ according to Lemma 1
- $n = 2 \cdot p^k$, ***p*>2 prime number and *k*>1** $\Rightarrow SS(n) \leq n$ according to Lemma 2.
- $n = 3 \cdot p^k$, **(*p*>3 prime number and *k*>1) or (*p*=2 and k>2)** $\Rightarrow SS(n) \leq n$ according to Remark 3.
- $n = 6 \cdot p^k$, ***p*>3 prime number and *k*≥1** $\Rightarrow SS(n) \leq n$ according to Lemma 3.
- **Otherwise** $\Rightarrow$ Let $n = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_s^{k_s}$ be the prime number decomposition of n with $p_1 < p_2 < ... < p_s$. We prove that there is a decomposition of $n=ab$, $(a,b)=1$ such that $S(a), S(b) \geq 4$. Let us select $a = p_s^{k_s}$ and $b = p_1^{k_1} \cdot p_2^{k_2} \cdot ... \cdot p_{s-1}^{k_{s-1}}$. It is not difficult to see that this decomposition satisfies the above conditions. The induction's hypotheses is applied for $a,b<n$ and the inequalities $SS(a) \leq a$ and $SS(b) \leq b$ are obtained. Finally, Proposition 3 gives $SS(n) = SS(a \cdot b) \leq SS(b) \cdot SS(a) \leq b \cdot a = n$.

We can conclude that the inequality $SS(n) \leq n-2$ holds for all the natural number $n \neq 12$.

♣

***Remark* 5.** The above analysis is necessary to be sure that the decomposition of $n=ab$, $(a,b)=1$, $S(a)$, $S(b) \geq 4$ exists.

Theorem 1 has some interesting consequences that are presented in the following. These establish new upper bounds for Smarandache's function.

***Consequence* 1.** If $n > 1$ is a natural number, then the following inequality

$$S(n) \leq n + 4 - 2 \cdot d(n) \quad (23)$$

holds.

Proof

The proof of this inequality is made by using Theorem 1.

Obviously, (23) is true for $n=p$ or $n=2p$, p prime number.

Let $n \neq 8, 12, 20$ be a natural number.

We have the following transformations:

$$n \geq SS(n) = \sum_{1 \neq d|n} S(d) = S(n) + \sum_{1,n \neq d|n} S(d) \geq$$

$$\geq S(n) + 2 \cdot \left| \left\{ d = \overline{1,n} \,|\, d \neq 1, n \wedge d \mid n \right\} \right| = S(n) + 2 \cdot (d(n) - 2) = S(n) + 2 \cdot d(n) - 4$$

Inequality (23) is also satisfied for n=8, 12, 20.

Therefore, the inequality $S(n) \leq n + 4 - 2 \cdot d(n)$ holds. ♣

Consequence 2. If $n > 1$ is a natural number, then the following inequality holds

$$S(n) \leq n + 4 - \min\{p \mid p \text{ is prime and } p|n\} \cdot d(n) \quad . \tag{24}$$

Proof

This proof is made similarly to the proof of the previous consequence by using the following strong inequality $S(d) \geq \min\{p \mid p \text{ is prime and } p|n\}$. ♣

3. Final Remark

Inequalities (23 - 24) give some generalisations of the well - known inequality $S(n) \leq n$. More important is the fact that these inequalities reflect. When n has many divisors, the value of $n + 4 - \min\{p \mid p \text{ is prime and } p|n\} \cdot d(n)$ is small, therefore the value of $S(n)$ is small as well according to Inequality (24). In spite of fact that Inequalities (23 - 24) reflect this situation, we could not say that the upper bounds are the lowest possible. Nevertheless, they offer a better upper bound than the inequality $S(n) \leq n$.

References

Cojocaru, I and Cojocaru, S (1997) On a Function in Number Theory, *Smarandache Notions Journal*, **8**, 164-169.

Ibstedt, H. (1997) *Surfing on the Ocean of Numbers - a few Smarandache Notions and Similar Topics*, Erhus University Press.

Tabirca, S. and Tabirca, T. (1997) Some upper bounds for Smarandache's function, *Smarandache Notions Journal*, **8**, 205-211.

Tabirca, S. and Tabirca, T. (1998) Two new functions in number theory and their applications at Smarandache's function, *Liberthas Mathematica*, **16**.

Smarandache, F (1980) A Function in Number Theory, *Analele Univ. Timisoara*, **XVIII**.

Smarandache's function applied to perfect numbers

Sebastián Martín Ruiz

5 August 1998

Smarandache's function may be defined as follows:
$S(n)$ = the smallest positive integer such that $S(n)!$ is divisible by n. [1]
In this article we are going to see that the value this function takes when n is a perfect number of the form $n = 2^{k-1} \cdot (2^k - 1)$, $p = 2^k - 1$ being a prime number.

Lemma 1 *Let $n = 2^i \cdot p$ when p is an odd prime number and i an integer such that:*

$$0 \leq i \leq E(\frac{p}{2}) + E(\frac{p}{2^2}) + E(\frac{p}{2^3}) + \cdots + E(\frac{p}{2^{E(\log_2 p)}}) = e_2(p!)$$

Where $e_2(p!)$ is the exponent of 2 in the prime number descomposition of $p!$.
$E(x)$ is the greatest integer less than or equal to x.
One has that $S(n) = p$.

Demonstration:

Given that $gcd(2^i, p) = 1$ (gcd = greatest common divisor) one has that $S(n) = max\{s(2^i), S(p)\} \geq S(p) = p$. Therefore $S(n) \geq p$.

If we prove that p! is divisible by n then one would have the equality.

$$p! = p_1^{e_{p_1}(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_s^{e_{p_s}(p!)}$$

where p_i is the i-th prime of the prime number descomposition of $p!$. It is clear that $p_1 = 2$, $p_s = p$, $e_{p_s}(p!) = 1$ for which:

$$p! = 2^{e_2(p!)} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)} \cdot p$$

From where one can deduce that:

$$\frac{p!}{n} = 2^{e_2(p!)-i} \cdot p_2^{e_{p_2}(p!)} \cdots p_{s-1}^{e_{p_{s-1}}(p!)}$$

is a positive integer since $e_2(p!) - i \geq 0$.

Therefore one has that $S(n) = p$

Proposition 1 *If n a perfec number of the form $n = 2^{k-1} \cdot (2^k - 1)$ with k a positive integer, $2^k - 1 = p$ prime, one has that $S(n) = p$.*

Demonstration:

For the Lemma it is sufficient to prove that $k - 1 \leq e_2(p!)$.

If we can prove that

$$k - 1 \leq 2^{k-1} - \frac{1}{2} \tag{1}$$

we will have proof of the proposition since:

$$k - 1 \leq 2^{k-1} - \frac{1}{2} = \frac{2^k - 1}{2} = \frac{p}{2}$$

As k-1 is an integer one has that $k - 1 \leq E(\frac{p}{2}) \leq e_2(p!)$

Proving (1) is the same as proving $k \leq 2^{k-1} + \frac{1}{2}$ at the same time, since k is integer, is equivalent to proving $k \leq 2^{k-1}$ (2).

In order to prove (2) we may consider the function: $f(x) = 2^{x-1} - x \quad x \in \mathbb{R}$. This function may be derived and its derivate is $f'(x) = 2^{x-1} ln\ 2 - 1$.

f will be increasing when $2^{x-1} ln\ 2 - 1 > 0$ resolving x:

$$x > 1 - \frac{ln\ (ln\ 2)}{ln\ 2} \simeq 1'5287$$

In particular f will be increasing $\forall\ x \geq 2$.

Therefore $\forall\ x \geq 2 \quad f(x) \geq f(2) = 0$ that is to say $2^{x-1} - x \geq 0 \quad \forall\ x \geq 2$ therefore

$$2^{k-1} \geq k \quad \forall\ k \geq 2 \quad integer$$

and thus is proved the proposition.

EXAMPLES:

$6 = 2 \cdot 3$	$S(6) = 3$
$28 = 2^2 \cdot 7$	$S(28) = 7$
$496 = 2^4 \cdot 31$	$S(496) = 31$
$8128 = 2^6 \cdot 127$	$S(8128) = 127$

References:

[1] C. Dumitrescu and R. Müller: *To Enjoy is a Permanent Component of Mathematics*. SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2,(1998) pp 21-26.

Author:

Sebastián Martín Ruiz. Avda, de Regla 43. CHIPIONA (CADIZ) 11550 SPAIN.

ON THE DIVERGENCE OF THE SMARANDACHE HARMONIC SERIES

Florian Luca

For any positive integer n let $S(n)$ be the minimal positive integer m such that $n \mid m!$. In [3], the authors showed that

$$\sum_{n\geq 1} \frac{1}{S(n)^2} \tag{1}$$

is divergent and attempted, with limited succes, to gain information about the behaviour of the partial sum

$$A(x) = \sum_{n\leq x} \frac{1}{S(n)^2}$$

by comparing it with both $\log x$ and $\log x + \log\log x$.

In this note we show that none of these two functions is a suitable candidate for the order of magnitude of $A(x)$.

Here is the result. For any $\delta > 0$ and $x \geq 1$ denote by

$$A_\delta(x) = \sum_{n\leq x} \frac{1}{S(n)^\delta}, \tag{2}$$

$$B_\delta(x) = \log A_\delta(x). \tag{3}$$

Then,

Theorem.

For any $\delta > 0$,

$$B_\delta(x) \geq \log 2 \cdot \frac{\log x}{\log\log x} - o\Big(\frac{\log x}{\log\log x}\Big). \tag{4}$$

What the above theorem basically says is that for fixed δ and for arbitrary $\epsilon > 0$, there exists some constant C (depending on both δ and ϵ), such that

$$A_\delta(x) > 2^{(1-\epsilon)\frac{\log x}{\log\log x}} \qquad \text{for } x > C. \tag{5}$$

Notice that equation (5) asserts that $A_\delta(x)$ grows much faster than any polynomial in $\log(x)$, so one certainly shouldn't try to approximate it by a linear in $\log x$.

The Proof.

In [1], we showed that

$$\sum_{n\geq 1} \frac{1}{S(n)^\delta} \tag{6}$$

diverges for all $\delta > 0$. Since the argument employed in the proof is relevant for our purposes, we reproduce it here.

Let $t \geq 1$ be an integer and $p_1 < p_2 < \dots < p_t$ be the first t prime numbers. Notice that any integer $n = p_t m$ where m is squarefree and all the prime factors of

m belong to $\{p_1, p_2, \ldots, p_{t-1}\}$ will certainly satisfy $S(n) = p_t$. Since there are at least 2^{t-1} such m's (the power of the set $\{p_1, \ldots, p_{t-1}\}$), it follows that series (6) is bounded below by

$$\sum_{t\geq 1} \frac{2^{t-1}}{p_t^{\delta}} = \sum_{t\geq 1} 2^{t-1-\delta \log_2 p_t}. \tag{7}$$

The argument ends noticing that since

$$\lim_{t\to\infty} \frac{p_t}{t \log t} = 1,$$

it follows that the exponent $t - 1 - \delta \log_2 p_t$ is always positive for t large enough. This proves the divergence of the series (6).

For the present theorem, the only new thing is the fact that we do not work with the whole series (6) but only with its partial sum $A_\delta(x)$. In particular, the parameter t from the above argument is precisely the maximal value of s for which $p_1 p_2 \ldots p_s \leq x$. In order to prove our theorem, we need to come up with a good lower bound on t.

We show that for all $\epsilon > 0$ one has

$$t > (1-\epsilon)\frac{\log x}{\log\log x} \tag{8}$$

provided that x is enough large. Assume that this is not so. It then follows that there exists some $\epsilon > 0$ such that

$$t < (1-\epsilon)\frac{\log x}{\log\log x} \tag{9}$$

for arbitrarily large values of x. Since t was the value of the maximal s such that $p_1 p_2 \ldots p_s \leq x$, it follows that

$$p_1 p_2 \ldots p_{t+1} > x. \tag{10}$$

From a formula in [2], it follows easily that

$$p_i \leq 2i \log i \qquad \text{for } i \geq 3. \tag{11}$$

It now follows, by taking logarithms in (10) and using (11), that

$$\log x < \sum_{i=1}^{t-1} \log p_i < C_1 + (t-1)\log 2 + \sum_{i=3}^{t+1} (\log i + \log\log i) <$$

$$C_1 + (t-1)\log 2 + \int_3^{t+2} (\log y + \log\log y)dy <$$

$$C_1 + (t-1)\log 2 + (t+2)(\log(t+2) + \log\log(t+2)), \tag{12}$$

where $C_1 = \log 6$. Since t can be arbitrarily large (because x is arbitrarily large), it follows that one can just work with

$$\log x < t(\log t + 2\log\log t). \tag{13}$$

Indeed, the amount $(t+2)(\log(t+2) + \log\log(t+2))$ appearing in the right hand side of (12) can be replaced by $t(\log t + \log\log t) + f(t)$ where $f(t) = 2\log t + 2\log\log t +$

$O(1)$ and then the sum of $f(t)$ with the linear term from the right hand side of (12) can certainly be bounded above by $t \log\log t$ for t large enough. Hence,

$$\log x < t(\log t + 2\log\log t). \tag{14}$$

Using inequality (9) to bound the factor t appearing in (14) in terms of x and the obvious inequality

$$t \leq (1-\epsilon)\frac{\log x}{\log\log x} < \log x$$

to bound the t's appearing inside the logs in (14), one gets

$$\log x < (1-\epsilon)\frac{\log x}{\log\log x}\Big(\log\log x + 2\log\log\log x\Big) = (1-\epsilon)\log x\Big(1 + \frac{2\log\log\log x}{\log\log x}\Big)$$

or, after some immediate simplifications,

$$\log\log x < \frac{2(1-\epsilon)}{\epsilon}\log\log\log x. \tag{15}$$

Since ϵ was fixed, it follows that inequality (15) cannot happen for arbitrarily large values of x. This proves that indeed (8) holds for any ϵ provided that x is large enough. We are now done. Indeed, going back to formula (7), it follows that

$$A_\delta(x) \geq 2^{t-1-\delta\log p_t}$$

or

$$B_\delta(x) = \log A_\delta(x) > \log 2(t - 1 - \delta\log p_t) > \log 2(t - 1 - \delta\log(2t\log t)), \tag{16}$$

where the last inequality in (16) follows from (11). By inequalities (8) and (16), we get

$$B_\delta(x) = t\log 2 - o(t) = \log 2(1-\epsilon)\frac{\log x}{\log\log x} - o(t).$$

Since ϵ could, in fact, be chosen arbitrarily small, we get

$$B_\delta(x) = \log 2\frac{\log x}{\log\log x} - o\Big(\frac{\log x}{\log\log x}\Big), \qquad 16)$$

which concludes the proof.

Remark.

We conjecture that the exact order of $B_\delta(x)$ is $\dfrac{\log x}{\log\log x} + O\Big(\dfrac{\log x}{(\log\log x)^2}\Big)$.

References

[1] F. Luca, "On the convergence of some series", preprint, 1998.

[2] J.B. Roser & L. Schoenfeld, "Approximate formulas for some functions of prime numbers", *Illinois J. of Math.* **6** (1962), pp. 64-94.

[3] S. Tabirca & T. Tabirca, "The convergence of Smarandache Harmonic Series", *SNJ* **9** (1998) No. 1-2, pp. 27-35.

A Generalisation of Euler's function

*Tatiana Tabirca** *Sabin Tabirca***

*Transilvania University of Brasov, Computer Science Department
*Bucks University College, Computing Department

The aim of this article is to propose a generalisation for Euler's function. This function is $\varphi : N \to N$ defined as follows $(\forall\, n \in N)\, \varphi(n) = \left|\{k = \overline{1,n} \mid (k,n) = 1\}\right|$. Perhaps, this is the most important function in number theory having many properties in number theory, combinatorics, *etc*. The main properties [Hardy & Wright, 1979] of this function are enumerated in the following:

$$(\forall\, a,b \in N)\,(a,b) = 1 \Rightarrow \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \text{ - the multiplicative property} \quad (1)$$

$$a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s} \Rightarrow \varphi(a) = a \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s}\right) \quad (2)$$

$$(\forall\, a \in N) \sum_{d|a} \varphi(d) = a. \quad (3)$$

More properties concerning this function can be found in [Hardy & Wright, 1979], [Jones & Jones, 1998] or [Rosen, 1993].

1. Euler's Function by order *k*

In the following, we shall see how this function is generalised such that the above properties are still kept. The way that will be used to introduce Euler's generalised function is from the function's formula to the function's properties.

***Definition* 1.** Euler's function by order $k \in N$ is $\varphi_k : N \to N$ defined by

$$\left(\forall a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s}\right) \varphi_k(a) = a^k \cdot \left(1 - \frac{1}{p_1^k}\right) \cdot \left(1 - \frac{1}{p_2^k}\right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s^k}\right).$$

***Remarks* 1.**

1. Let us assume that $\varphi_k(1) = 1$.

2. Euler function by order 1 is Euler's function. Obviously, Euler's function by order 0 is the constant function 1.

In the following, the main properties of Euler's function by order k are proposed.

***Theorem* 1.** Euler's function by order k is multiplicative

$$(\forall\, a,b \in N)(a,b)=1 \Rightarrow \varphi_k(a\cdot b)=\varphi_k(a)\cdot\varphi_k(b). \tag{4}$$

Proof

This proof is obvious from the definition. ♣

***Theorem* 2.** $(\forall\, a \in N)\sum_{d|a}\varphi_k(d)=a^k$. (5)

Proof

The function $\overline{\varphi_k}(a)=\sum_{d|a}\varphi_k(d)$ is multiplicative because φ_k is a multiplicative function.

If $a=p^m$, then the following transformation proves (5)

$$\overline{\varphi_k}(a)=\sum_{d|p^m}\varphi_k(d)=\sum_{i=0}^{m}\varphi_k(p^i)=1+\sum_{i=1}^{m}p^{k\cdot i}\left(1-\frac{1}{p^k}\right)=$$

$$=1+\sum_{i=1}^{m}\left(p^{k\cdot i}-p^{k\cdot(i-1)}\right)=1+p^{k\cdot m}-1=a^k$$

If $a=p_1^{m_1}\cdot p_2^{m_2}\cdot\ldots\cdot p_s^{m_s}$ then the multiplicative property is applied as follows:

$$\overline{\varphi_k}(a)=\overline{\varphi_k}(p_1^{m_1})\cdot\overline{\varphi_k}(p_2^{m_2})\cdot\ldots\cdot\overline{\varphi_k}(p_s^{m_s})=p_1^{k\cdot m_1}\cdot p_2^{k\cdot m_2}\cdot\ldots\cdot p_s^{k\cdot m_s}=a^k.$$ ♣

***Definition* 2.** A natural number n is said to be k-power free if there is not a prim number p such that $p^k \mid n$.

***Remarks* 2.**

1. There is not a 0-power free number.
2. Assume that 1 is the only 1-power free number.

The combinatorial property of Euler's function by order k is given by the following theorem. This property is introduced by using the k-power free notion.

***Theorem* 3.** $(\forall\, n \in N)\, \varphi_k(n) = \left|\left\{ i = \overline{1, n^k} \mid (i, n^k) \text{ is } k \text{ - power free} \right\}\right|$ (6)

Proof

This proof is made using the Inclusion-Exclusion theorem.

Let $a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots \cdot p_s^{m_s}$ be the prime number decomposition of a.

If $d|n$, then the set $S_d = \left\{ i = \overline{1, a^k} \mid d^k \mid i \right\}$ contains all the numbers that have the divisor d^k. This set satisfies the following properties:

$$S_d = \left\{ d^k, 2 \cdot d^k, \ldots, \frac{a^k}{d^k} \cdot d^k \right\} \Rightarrow |S_d| = \frac{a^k}{d^k} \tag{7}$$

$$1 \le j_1 < j_2 \le n \Rightarrow S_{p_{j_1}} \cap S_{p_{j_2}} = S_{p_{j_1} \cdot p_{j_2}}$$

(8)

$$\left\{ i = \overline{1, a^k} \mid (i, a^k) \text{ is } k \text{ - power free} \right\} = \left\{ i = \overline{1, a^k} \right\} - \left(S_{p_1} \cap S_{p_2} \cap \ldots \cap S_{p_s} \right). \tag{9}$$

The Inclusion-Exclusion theorem and (7-9) give the following transformations:

$$\left|\left\{ i = \overline{1, a^k} \mid (i, a^k) \text{ is } k \text{ - power free} \right\}\right| = a^k - \left| \left(S_{p_1} \cap S_{p_2} \cap \ldots \cap S_{p_s} \right) \right| =$$

$$= a^k - \sum_{j=1}^{s} \left| S_{p_j} \right| + \sum_{1 \le j_1 < j_2 \le n} \left| S_{p_{j_1}} \cap S_{p_{j_2}} \right| - \ldots + (-1)^{s+1} \sum_{1 \le j_1 < j_2 < \ldots < j_s \le n} \left| S_{p_{j_1}} \cap S_{p_{j_2}} \cap \ldots \cap S_{p_{j_s}} \right| =$$

$$= a^k - \sum_{j=1}^{s} \left| S_{p_j} \right| + \sum_{1 \le j_1 < j_2 \le n} \left| S_{p_{j_1} \cdot p_{j_2}} \right| - \ldots + (-1)^{s+1} \sum_{1 \le j_1 < j_2 < \ldots < j_s \le n} \left| S_{p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_2}} \right| =$$

$$= a^k - \sum_{j=1}^{s} \frac{a^k}{p_j^k} + \sum_{1 \le j_1 < j_2 \le n} \frac{a^k}{\left(p_{j_1} \cdot p_{j_2} \right)^k} - \ldots + (-1)^{s+1} \sum_{1 \le j_1 < j_2 < \ldots < j_s \le n} \frac{a^k}{\left(p_{j_1} \cdot p_{j_2} \cdot \ldots \cdot p_{j_2} \right)^k} =$$

$$= a^k \cdot \left(1 - \frac{1}{p_1^k} \right) \cdot \left(1 - \frac{1}{p_2^k} \right) \cdot \ldots \cdot \left(1 - \frac{1}{p_s^k} \right) = \varphi_k(a)$$

Therefore, the equation (6) holds. ♣

3. Conclusion

Euler's function by order k represents a successful way to generalise Euler's function. Firstly, because the main properties of Euler's function (1-3) have been extended for Euler's function by order k. Secondly and more important, because a combinatorial property has been found for this generalised function. Obviously, many other properties can be deduced for Euler's function by order k. Unfortunately, a similar property with Euler's theorem $a^{\varphi(n)} = 1 \bmod n$ has not been found so far.

References

Hardy, G. H. and Wright, E. M. (1979) *Introduction to the Theory of Numbers*, (5th ed.) Oxford University Press, Oxford.

Jones, G. A. and Jones, M. J. (1998) *Elementary Number Theory*, Springer-Verlag, New York.

Rosen, K (1993) *Elementary Number Theory and its Application*, Addison-Wesley, New York.

A result obtained using Smarandache Function

Sebastián Martín Ruiz

21 November 1998

Smarandache Function is defined as followed:
$S(m)$ = The smallest positive integer so that $S(m)!$ is divisible by m. [1]
Let's see the value which such function takes for $m = p^{p^n}$ with n integer, $n \geq 2$ and p prime number. To do so a Lemma is required.

Lemma 1 $\forall \quad m, n \in \mathbf{N} \quad m, n \geq 2$

$$m^n = E\left[\frac{m^{n+1} - m^n + m}{m}\right] + E\left[\frac{m^{n+1} - m^n + m}{m^2}\right] + \cdots + E\left[\frac{m^{n+1} - m^n + m}{m^{E[\log_m(m^{n+1} - m^n + m)]}}\right]$$

Where $E(x)$ gives the greatest integer less than or equal to x.

Demonstration:

Let's see in the first place the value taken by $E[\log_m(m^{n+1} - m^n + m)]$.

If $n \geq 2$: $m^{m+1} - m^n + m < m^{n+1}$ and therefore $\log_m(m^{n+1} - m^n + m) < \log_m m^{n+1} = n + 1$. As a result $E[\log_m(m^{n+1} - m^n + m)] < n + 1$.

And if $m \geq 2$: $mm^n \geq 2m^n \Rightarrow m^{n+1} \geq 2m^n \Rightarrow m^{n+1} + m \geq 2m^n \Rightarrow m^{n+1} - m^n + m \geq m^n \Rightarrow \log_m(m^{n+1} - m^n + m) \geq \log_m m^n = n \Rightarrow E[\log_m(m^{n+1} - m^n + m)] \geq n$

As a result: $n \leq E[\log_m(m^{n+1} - m^n + m)] < n + 1$ therefore:

$$E[\log_m(m^{n+1} - m^n + m)] = n \quad if\ n, m \geq 2$$

Now let's see the value which it takes for $1 \leq k \leq n$: $E\left[\frac{m^{n+1} - m^n + m}{m^k}\right]$

$$E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = E\left[m^{n+1-k} - m^{n-k} + \frac{1}{m^{k-1}}\right]$$

If $k = 1$: $E\left[\frac{m^{n+1} - m^n + m}{m}\right] = m^n - m^{n-1} + 1$

If $1 < k \leq n$: $E\left[\frac{m^{n+1} - m^n + m}{m^k}\right] = m^{n+1-k} - m^{n-k}$

Let's see what is the value of the sum:

$$\begin{array}{lcccccccc}
k=1 & m^n & -m^{n-1} & \cdots & \cdots & \cdots & \cdots & +1 \\
k=2 & & m^{n-1} & -m^{n-2} & & & & \\
k=3 & & & m^{n-2} & -m^{n-3} & & & \\
\vdots & & & & \vdots & & & \\
k=n-1 & & & & & m^2 & -m & \\
k=n & & & & & & m & -1
\end{array}$$

Therefore:

$$\sum_{k=1}^{n} E\left[\frac{m^{n+1}-m^n+m}{m^k}\right] = m^n \quad m,n \geq 2$$

Proposition 1 $\forall\, p$ *prime number* $\forall n \geq 2$:

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

Demonstration:

Having $e_p(k)$ = exponent of the prime number p in the prime numbers descomposition of k.

We get:

$$e_p(k!) = E(\frac{k}{p}) + E(\frac{k}{p^2}) + E(\frac{k}{p^3}) + \cdots + E(\frac{k}{p^{E(\log_p k)}})$$

And using the Lemma we have:

$$e_p[(p^{n+1}-p^n+p)!] = E\left[\frac{p^{n+1}-p^n+p}{p}\right] + E\left[\frac{p^{n+1}-p^n+p}{p^2}\right] + \cdots + E\left[\frac{p^{n+1}-p^n+p}{m^{E[\log_p(p^{n+1}-p^n+p)]}}\right] = p^n$$

Therefore:

$$\frac{(p^{n+1}-p^n+p)!}{p^{p^n}} \in \mathbf{N} \quad \textit{and} \quad \frac{(p^{n+1}-p^n+p-1)!}{p^{p^n}} \notin \mathbf{N}$$

And:

$$S(p^{p^n}) = p^{n+1} - p^n + p$$

References:

[1] C. Dumitrescu and R. Müller: *To Enjoy is a Permanent Component of Mathematics.* SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2.(1998) pp 21-26.

Author:

Sebastián Martín Ruiz. Avda. de Regla 43. CHIPIONA (CADIZ) 11550 SPAIN.

On an inequality for the Smarandache function

J. Sándor

Babeş-Bolyai University, 3400 Cluj-Napoca, Romania

1. In paper [2] the author proved among others the inequality $S(ab) \leq aS(b)$ for all a, b positive integers. This was refined to

$$S(ab) \leq S(a) + S(b) \tag{1}$$

in [1]. Our aim is to show that certain results from our recent paper [3] can be obtained in a simpler way from a generalization of relation (1). On the other hand, by the method of Le [1] we can deduce similar, more complicated inequalities of type (1).

2. By mathematical induction we have from (1) immediately:

$$S(a_1 a_2 \dots a_n) \leq S(a_1) + S(a_2) + \dots + S(a_n) \tag{2}$$

for all integers $a_i \geq 1$ $(i = 1, \dots, n)$. When $a_1 = \dots = a_n = n$ we obtain

$$S(a^n) \leq nS(a). \tag{3}$$

For three applications of this inequality, remark that

$$S((m!)^n) \leq nS(m!) = nm \tag{4}$$

since $S(m!) = m$. This is inequality 3) part 1. from [3]. By the same way, $S((n!)^{(n-1)!}) \leq (n-1)!S(n!) = (n-1)!n = n!$, i.e.

$$S((n!)^{(n-1)!}) \leq n! \tag{5}$$

Inequality (5) has been obtained in [3] by other arguments (see 4) part 1.).

Finally, by $S(n^2) \le 2S(n) \le n$ for n even (see [3], inequality 1), $n > 4$, we have obtained a refinement of $S(n^2) \le n$:

$$S(n^2) \le 2S(n) \le n \tag{6}$$

for $n > 4$, even.

3. Let m be a divisor of n, i.e. $n = km$. Then (1) gives $S(n) = S(km) \le S(m) + S(k)$, so we obtain:

If $m|n$, then

$$S(n) - S(m) \le S\left(\frac{n}{m}\right). \tag{7}$$

As an application of (7), let $d(n)$ be the number of divisors of n. Since $\prod_{k|n} k = n^{d(n)/2}$, and $\prod_{k\le n} k = n!$ (see [3]), and by $\prod_{k|n} k | \prod_{k\le n} k$, from (7) we can deduce that

$$S(n^{d(n)/2}) + S(n!/n^{d(n)/2}) \ge n. \tag{8}$$

This improves our relation (10) from [3].

4. Let $S(a) = u$, $S(b) = v$. Then $b|v!$ and $u!|x(x-1)\dots(x-u+1)$ for all integers $x \ge u$. But from $a|u!$ we have $a|x(x-1)\dots(x-u+1)$ for all $x \ge u$. Let $x = u+v+k$ $(k \ge 1)$. Then, clearly $ab(v+1)\dots(v+k)|(u+v+k)!$, so we have $S[ab(v+1)\dots(v+k)] \le u+v+k$. Here $v = S(b)$, so we have obtained that

$$S[ab(S(b)+1)\dots(S(b)+k)] \le S(a) + S(b) + k. \tag{9}$$

For example, for $k = 1$ one has

$$S[ab(S(b)+1)] \le S(a) + S(b) + 1. \tag{10}$$

This is not a consequence of (2) for $n = 3$, since $S[S(b)+1]$ may be much larger than 1.

References

[1] M. Le, *An inequality concerning the Smarandache function*, Smarandache Notions J., vol. **9**(1998), 124-125.

[2] J. Sándor, *On certain inequalities involving the Smarandache function*, Smarandache Notions J., vol. **7**(1996), 3-6.

[3] J. Sándor, *On certain new inequalities and limits for the Smarandache function*, Smarandache Notions J., vol. **9**(1998), 63-69.

ON A SERIES INVOLVING $S(1) \cdot S(2) \ldots \cdot S(n)$

Florian Luca

For any positive integer n let $S(n)$ be the minimal positive integer m such that $n \mid m!$. It is known that for any $\alpha > 0$, the series

$$\sum_{n \geq 1} \frac{n^\alpha}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)} \tag{1}$$

is convergent, although we do not know who was the first to prove the above statement (for example, the authors of [4] credit the paper [1] appeared in 1997, while the result appears also as Proposition 1.6.12 in [2] which was written in 1996).

In this paper we show that, in fact:

Theorem.

The series

$$\sum_{n \geq 1} \frac{x^n}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)} \tag{2}$$

converges absolutely for every x.

Proof

Write

$$a_n = \frac{|x|^n}{S(1) \cdot S(2) \cdot \ldots \cdot S(n)}. \tag{3}$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{|x|}{S(n+1)}. \tag{4}$$

But for $|x|$ fixed, the ratio $|x|/S(n+1)$ tends to zero. Indeed, to see this, choose any positive real number m, and let $n_m = \lfloor m|x| + 1 \rfloor!$. When $n > n_m$, it follows that $S(n+1) > \lfloor m|x| + 1 \rfloor > m|x|$, or $S(n+1)/|x| > m$. Since m was arbitrary, it follows that the sequence $S(n+1)/|x|$ tends to infinity.

Remarks.

1. The convergence of (2) is certainly better than the convergence of (1). Indeed, if one fixes any $x > 1$ and any α, then certainly $x^n > n^\alpha$ for n large enough.

2. The convergence of (2) combined with the root test imply that

$$(S(1) \cdot S(2) \cdot \ldots \cdot S(n))^{1/n}$$

diverges to infinity. This is equivalent to the fact that the average function of the logs of S, namely

$$LS(x) = \frac{1}{x} \sum_{n \leq x} \log S(n) \qquad \text{for } x \geq 1$$

tends to infinity with x. It would be of interest to study the order of magnitude of the function $LS(x)$. We conjecture that

$$LS(x) = \log x - \log\log x + O(1). \tag{5}$$

The fact that $LS(x)$ cannot be larger than what shows up in the right side of (5) follows from a result from [3]. Indeed, in [3], we showed that

$$A(x) = \frac{1}{x}\sum_{n\leq x} S(n) < 2\frac{x}{\log x} \qquad \text{for } x \geq 64. \tag{6}$$

Now the fact that $LS(x) - \log x + \log\log x$ is bounded above follows from (6) and from Jensen's inequality for the log function (or the logarithmic form of the AGM inequality). It seems to be considerably harder to prove that $LS(x)-\log x+\log\log x$ is bounded below.

3. As a fun application we mention that for every integer $k \geq 1$, the series

$$\sum_{n\geq 1} \binom{n}{k} \cdot \frac{x^n}{S(1)\cdot S(2)\cdot \ldots \cdot S(n)} \tag{7}$$

is absolutely convergent. Indeed, it is a straightforward computation to verify that if one denotes by $C(x)$ the sum of the series (2), then the series (7) is precisely

$$\frac{x^k}{k!}\cdot\frac{d^kC}{dx^k}. \tag{8}$$

When $k = x = 1$ series (7) becomes precisely series (1) for $\alpha = 1$.

4. It could be of interest to study the rationality of (2) for integer values of x. Indeed, if the function S is replaced with the identity in formula (2), then one obtains the more familiar e^x whose value is irrational (in fact, transcendental) at all integer values of x. Is that still true for series (2)?

References

[1] I. Cojocaru & S. Cojocaru, "On a function in number theory", *SNJ* **8** (1997), pp. 164-169.

[2] C. Dumitrescu & V. Seleacu, "The Smarandache function", Erhus U. Press, 1996.

[3] F. Luca, "The average Smarandache function", preprint 1998.

[4] S. Tabirca & T. Tabirca, "The convergence of Smarandache Harmonic Series", *SNJ* **9** (1998) No. 1-2, pp. 27-35.

A Congruence with Smarandache's Function

Sebastián Martín Ruiz

25 September 1998

Smarandache's function is defined thus:
$S(n) =$ is the smallest integer such that $S(n)!$ is divisible by n. [1]
In this article we are going to look at the value that has $S(2^k - 1) \pmod k$ for all k integer, $2 \leq k \leq 97$.

One can observe in the following table that gives the continuation $S(2^k - 1) \equiv 1 \pmod k$ in the majority of cases, there are only 4 exeptions for $2 \leq k \leq 97$.

k	$S(2^k - 1)$	$S(2^k - 1) \, (mod\ k)$
2	3	1
3	7	1
4	5	1
5	31	1
6	7	1
7	127	1
8	17	1
9	73	1
10	31	1
11	89	1
12	13	1
13	8191	1
14	127	1
15	151	1
16	257	1
17	131071	1
18	73	1
19	524287	1

k	$S(2^k-1)$	$S(2^k-1)\ (mod\ k)$	k	$S(2^k-1)$	$S(2^k-1)\ (mod\ k)$
20	41	1	59	3203431780337	1
21	337	1	60	1321	1
22	683	1	61	2305843009213693951	1
23	178481	1	62	2147483647	1
24	241	1	63	649657	1
25	1801	1	64	6700417	1
26	8191	1	65	145295143558111	1
27	262657	1	66	599479	1
28	127	**15**	67	761838257287	1
29	2089	1	68	131071	**35**
30	331	1	69	10052678938039	1
31	2147483647	1	70	122921	1
32	65537	1	71	212885833	1
33	599479	1	72	38737	1
34	131071	1	73	9361973132609	1
35	122921	1	74	616318177	1
36	109	1	75	10567201	1
37	616318177	1	76	525313	1
38	524287	1	77	581283643249112959	1
39	121369	1	78	22366891	1
40	61681	1	79	1113491139767	1
41	164511353	1	80	4278255361	1
42	5419	1	81	97685839	1
43	2099863	1	82	8831418697	1
44	2113	1	83	57912614113275649087721	1
45	23311	1	84	14449	1
46	2796203	1	85	9520972806333758431	1
47	13264529	1	86	2932031007403	1
48	673	1	87	9857737155463	1
49	4432676798593	1	88	2931542417	1
50	4051	1	89	618970019642690137449562111	1
51	131071	1	90	18837001	1
52	8191	**27**	91	23140471537	1
53	20394401	1	92	2796203	**47**
54	262657	1	93	658812288653553079	1
55	201961	1	94	165768537521	1
56	15790321	1	95	30327152671	1
57	1212847	1	96	22253377	1
58	3033169	1	97	13842607235828485645766393	1

One can see from the table that there are only 4 exeptions for $2 \leq k \leq 97$.

We can see in detail the 4 exeptions in a table:

$k = 28 = 2^2 \cdot 7$	$S(2^{28} - 1) \equiv 15 \ (mod\ 28)$
$k = 52 = 2^2 \cdot 13$	$S(2^{52} - 1) \equiv 27 \ (mod\ 52)$
$k = 68 = 2^2 \cdot 17$	$S(2^{68} - 1) \equiv 35 \ (mod\ 68)$
$k = 92 = 2^2 \cdot 23$	$S(2^{92} - 1) \equiv 47 \ (mod\ 92)$

One can observe in these 4 cases that $k = 2^2 \cdot p$ with p prime and moreover $S(2^k - 1) \equiv \frac{k}{2} + 1 \ (mod\ k)$

Unsolved Question:

One can obtain a general formula that gives us, in function of k the value $S(2^k - 1) \ (mod\ k)$ for all pisitive integer values of k ?.

References:

[1] SMARANDACHE NOTIONS JOURNAL Vol. 9, No. 1-2,(1998) pp 21-26.

Sebastián Martín Ruiz. Avda, de Regla 43. CHIPIONA (CADIZ) 11550 SPAIN.

An Integer as a Sum of Consecutive Integers

Henry Ibstedt

Abstract: This is a simple study of expressions of positive integers as sums of consecutive integers. In the first part proof is given for the fact that N can be expressed in exactly d(L)-1 ways as a sum of consecutive integers, L is the largest odd factor of N and d(L) is the number of divisors of L. In the second part answer is given to the question: Which is the smallest integer that can be expressed as a sum of consecutive integers in n ways.

Introduction: There is a remarkable similarity between the four definitions given below. The first is the well known Smarandache Function. The second function was defined by K. Kashihara and was elaborated on in his book *Comments and Topics on Smarandache Notions and Problems*[1]. This function and the Smarandache Ceil Function were also examined in the author's book *Surfing on the Ocean of Numbers*[2]. These three functions have in common that they aim to answer the question **which is the smallest positive integer N which possesses a certain property pertaining to a given integer n.** It is possible to pose a large number of questions of this nature.

1. **The Smarandache Function S(n):**
   S(n)=N where N is the smallest positive integer which divides n!.

2. **The Pseudo-Smarandache Function Z(n):**
   Z(n)=N where N is the smallest positive integer such that 1+2+...+N is divisible by n.

3. **The Smarandache Ceil Function of order k, $S_k(n)$:**
   $S_k(n)=N$ where N is he smallest positive integer for which n divides N^k

4. **The n-way consecutive integer representation R(n):**
   R(n)=N where N is the smallest positive integer which can be represented as a sum of consecutive integer is n ways.

There may be many positive integers which can be represented as a sum of positive integers in n distinct ways - but which is the smallest of them? This article gives the answer to this question. In the study of R(n) it is found that the arithmetic function d(n), the number of divisors of n, plays an important role.

[1] Erhus University Press, 1996
[2] Erhus University Press, 1997

Question 1: In how many ways n can a given positive integer N be expressed as the sum of consecutive positive integers?

Let the first term in a sequence of consecutive integers be Q and the number terms in the sequence be M. We have N=Q+(Q+1)+ ... +(Q+M-1) where M>1.

$$N = \frac{M(2Q+M-1)}{2} \tag{1}$$

or

$$Q = \frac{N}{M} - \frac{M-1}{2} \tag{2}$$

For a given positive integer N the number of sequences n is equal to the number of positive integer solutions to (2) in respect of Q. Let us write $N=L\cdot 2^s$ and $M=m\cdot 2^k$ where L and m are odd integers. Furthermore we express L as a product of any of its factors $L=m_1 m_2$. We will now consider the following cases:

1. s=0, k=0
2. s=0, k≠0
3. s≠0, k=0
4. s≠0, k≠0

Case 1. s=0, k=0.

Equation (2) takes the form

$$Q = \frac{m_1 m_2}{m} - \frac{m-1}{2} \tag{3}$$

Obviously we must have m≠1 and m≠L (=N).

For $m=m_1$ we have Q>0 when $m_2-(m_1-1)/2>0$ or $m_1<2m_2+1$. Since m_1 and m_2 are odd, the latter inequality is equivalent to $m_1<2m_2$ or, since $m_2=N/m_1$, $m_1 < \sqrt{2N}$.
We conclude that a factor m (≠1 and ≠N) of N (odd) for which $m < \sqrt{2N}$ gives a solution for Q when M=m is inserted in equation (2).

Case 2. s=0, k≠0.

Since N is odd we see form (2) that we must have k=1. With M=2m equation (2) takes the form

$$Q = \frac{m_1 m_2}{2m} - \frac{2m-1}{2} \tag{4}$$

For m=1 (M=2) we find Q=(N-1)2 which corresponds to the obvious solution $\frac{N-1}{2}+\frac{N+1}{2}=N$.

Since we can have no solution for m=N we now consider $m=m_2$ ($\neq 1$, $\neq N$). We find $Q=(m_1-2m_2+1)/2$. Q>0 when $m_1>2m_2-1$ or, since m_1 and m_2 are odd, $m_1>2m_2$ Since $m_1m_2=N$, $m_2=N/m_1$ we find $m>\sqrt{2N}$.

We conclude that a factor m ($\neq 1$ and $\neq N$) of N (odd) for which $m>\sqrt{2N}$ gives a solution for Q when M=2m is inserted in equation (2).

The number of divisors of N is known as the function d(N). Since all factors of N except 1 and N provide solutions to (2) while M=2, which is not a factor of N, also provides a solution (2) we find that the number of solutions n to (2) when N is odd is

$$n=d(N)-1 \tag{5}$$

Case 3. $s\neq 0$, $k=0$.

Equation (2) takes the form

$$Q=\frac{1}{2}\left(\frac{m_1m_2}{m}2^{s+1}-m+1\right) \tag{6}$$

$Q\geq 1$ requires $m^2<L\cdot 2^{s+1}$. We distinguish three cases:

Case 3.1.	$k=0$, $m=1$.	There is no solution.
Case 3.2.	$k=0$, $m=m_1$.	$Q\geq 1$ for $m_1<m_2 2^{s+1}$ with a solution for Q when $M=m_1$.
Case 3.3.	$k=0$, $m=m_1m_2$.	$Q\geq 1$ for $L<2^{s+1}$ with a solution for Q when M=L.

Case 4. $s\neq 0$, $k\neq 0$.

Equation (2) takes the form

$$Q=\frac{1}{2}\left(\frac{m_1m_2}{m}2^{s-k+1}-m\cdot 2^k+1\right) \tag{7}$$

Q is an integer if and only if m divides L and $2^{s-k+1}=1$. The latter gives $k=s+1$. $Q\geq 1$ gives

$$Q=\frac{1}{2}\left(\frac{m_1m_2}{m}+1\right)-m\cdot 2^s\geq 1 \tag{8}$$

Again we distinguish three cases:

Case 4.1.	$k=s+1$, $m=1$.	$Q\geq 1$ for $L>2^{s+1}$ with a solution for Q when $M=2^{s+1}$
Case 4.2.	$k=s+1$, $m=m_2$	$Q\geq 1$ for $m_1>m_2 2^{s+1}$ with a solution for Q when $M=m_2 2^{s+1}$
Case 4.3.	$k=s+1$, $m=L$	$Q\geq 1$ for $1-L\cdot 2^s\geq 1$. No solution

Since all factors of L except 1 provide solutions to (2) we find that the number of solutions n to (2) when N is even is

$$n=d(L)-1 \tag{9}$$

Conclusions:

- The number of sequences of consecutive positive integers by which a positive integer $N=L\cdot 2^s$ where $L\equiv 1 \pmod 2$ can be represented is $n=d(L)-1$.
- We see that the number of integer sequences is the same for $N=2^sL$ and $N=L$ no matter how large we make s.
- When $L<2^s$ the values of M which produce integer values of Q are odd, i.e. N can in this case only be represented by sequences of consecutive integers with an odd number of terms.
- There are solutions for all positive integers L except for L=1, which means that $N=2^s$ are the only positive integers which cannot be expressed as the sum of consecutive integers.
- $N=P\cdot 2^s$ has only one representation which has a different number of terms (<p) for different s until $2^{s+1}>P$ when the number of terms will be p and remain constant for all larger s.

A few examples are given in table 1.

Table 1. The number of sequences for L=105 is 7 and is independent of s in $N=L\cdot 2^s$.

N=105	s=0	N=210	s=1	N=3360	s=5 $L>2^{s+1}$	N=6720	s=6 $L<2^{s+1}$
Q	M	Q	M	Q	M	Q	M
34	3	69	3	1119	3	2239	3
19	5	40	5	670	5	1342	5
12	7	27	7	477	7	957	7
1	14	7	15	217	15	441	15
6	10	1	20	150	21	310	21
15	6	12	12	79	35	175	35
52	2	51	4	21	64	12	105

Question 2: Which is the smallest positive integer N which can be represented as a sum of consecutive positive integers in n different ways.

We can now construct the smallest positive integer R(n)=N which can be represented in n ways as the sum of consecutive integers. As we have already seen this smallest integer is necessarily odd and satisfies n=d(N)-1.

With the representation $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_j^{\alpha_j}$ we have

$$d(N)=(\alpha_1+1)(\alpha_2+1)...(\alpha_j+1)$$

or

$$n+1=(\alpha_1+1)(\alpha_2+1)...(\alpha_j+1) \qquad (10)$$

The first step is therefore to factorize n+1 and arrange the factors (α_1+1), (α_2+1) ... (α_j+1) in descending order. Let us first assume that $\alpha_1>\alpha_2> ... >\alpha_j$ then, remembering that N must be odd, the smallest N with the largest number of divisors is

$$R(n)= N = 3^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3} ... p_j^{\alpha_j}$$

where the primes are assigned to the exponents in ascending order starting with $p_1=3$. Every factor in (10) corresponds to a different prime even if there are factors which are equal.

Example: $n = 269$
$n+1= 2\cdot3^3\cdot5 = 5 \cdot3 \cdot3 \cdot3 \cdot2$
$R(n)= 3^4\cdot5^2\cdot7^2\cdot11^2\cdot13=156080925$

When n is even it is seen from (10) that α_1, α_2, ... α_j must all be even. In other words the smallest positive integer which can be represented as a sum of consecutive integers in a given number of ways must be a square. It is therefore not surprising that even values of n in general generate larger smallest N than odd values of n. For example, the smallest integer that can be represented as a sum of integers in 100 ways is $N=3^{100}$, which is a 48-digit integer, while the smallest integer that can be expressed as a sum of integer in 99 ways is only a 7-digit integer, namely 3898125.

Conclusions:

- 3 is always a factor of the smallest integer that can be represented as a sum of consecutive integers in n ways.
- The smallest positive integer which can be represented as a sum of consecutive integers in given even number of ways must be a square.

Table 2. The smallest integer R(n) which can be represented in n ways as a sum of consecutive positive integers.

n	R(n)	R(n) in factor form
1	3	3
2	9	3^2
3	15	$3 \cdot 5$
4	81	3^4
5	45	$3^2 \cdot 5$
6	729	3^6
7	105	$3 \cdot 5 \cdot 7$
8	225	$3^2 5^2$
9	405	$3^4 5$
10	59049	3^{10}
11	315	$3^2 5 \cdot 7$
12	531441	3^{12}

A NEW INEQUALITY FOR THE SMARANDACHE FUNCTION

Mihaly Bencze
Str. Hărmanului 6
2212 Săcele 3, jud. Brasov
Romania.

Theorem. Let $S(m)=\min\{k\in N: m|k!\}$ be the Smarandache Function, and $a_k, b_k \in N^*$ $(k=1,2,\ldots,n)$, then we have the following inequality

$$S\left(\prod_{k=1}^{n}(a_k!)^{b_k}\right) \leq \sum_{k=1}^{n} a_k b_k$$

Proof:

$$\frac{\sum_{k=1}^{n}(a_k b_k)!}{\prod_{k=1}^{n}(a_k!)^{b_k}} = \frac{\sum_{k=1}^{n}(a_k b_k)!}{\prod_{k=1}^{n}(a_k b_k)!} * \frac{\prod_{k=1}^{n}(a_k b_k)!}{\prod_{k=1}^{n}(a_k!)^{b_k}} =$$

$$\binom{a_1 b_1 + a_2 b_2 + \ldots + a_m b_m}{a_1 b_1}\binom{a_2 b_2 + \ldots + a_m b_m}{a_2 b_2}\ldots\binom{a_{m-1} b_{m-1} + a_m b_m}{a_{m-1} b_{m-1}}$$

$$\left(\prod_{k=1}^{n}\binom{a_k b_k}{a_k}\ldots\binom{3a_k}{a_k}\binom{2a_k}{a_k}\right) \in N^*$$

From this result

$$S\left(\prod_{k=1}^{n}(a_k!)^{b_k}\right) \leq \sum_{k=1}^{n} a_k b_k$$

A FORMULA OF THE SMARANDACHE FUNCTION

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R. China.

Abstract. In this paper we give a formula expressing the Smarandache function S(n) by means of n without using the factorization of n.

For any positive integer n, let S (n) denote the Smarandache function of n. Then we have

$$(1) \qquad S(n) = \min\{ a \mid a \in N, \; n \mid a! \},$$

(See [1]). In this paper we give a formula of S(n) without using the factorization of n as follows:

Theorem. For any positive integer n, we have

$$(1) \qquad S(n) = n+1 - \left[\sum_{k=1}^{n} n^{-(n \sin (k! \pi / n))^2} \right]$$

Proof. Let a = S(n). It is an obvious fact that $1 \le a \le n$. We see from (1) that

$$(2) \qquad n \mid k!, \quad k = a, a+1, \ldots, n.$$

It implies that

$$(4) \qquad n^{-(n \sin (k!\,\pi / n))^2} = n^0 = 1, \qquad k = a,\ a+1, \dots, n.$$

On the other hand, since $n \nmid k!$ for $k = 1, \dots, a-1$, we have $\sin (k!\pi/n) \neq 0$ and

$$(5) \qquad \left(n \sin \frac{k!\,\pi}{n}\right)^2 \geq \left(n \sin \frac{\pi}{n}\right)^2 > 1, \quad k = 1, \dots, a-1.$$

Hense, by (5), we get

$$(6) \qquad 0 < n^{-(n \sin (k!\,\pi/ n))^2} < 1/n, \quad k = 1, \dots, a-1.$$

Therefore, by (4) and (6), we obtain

$$(7) \qquad n+1-a < \sum_{k=1}^{n} n^{-(n \sin (k!\,\pi/n))^2} < n+1-a+(a-1)/n < n+2-a.$$

Thus, by (7), we get (1) immediately. The theorem is proved.

Reference

1. F Smarandache, A function in the number theory, Smarandache function J. 1 (1990), No.1, 3 - 17.

ON THE DIOPHANTINE EQUATION S(n) = n

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R. China.

Abstract. Let S(n) denote the Smarandache function of n. In this paper we prove that $S(n) = n$ if and only if $n = 1, 4$ or p, where p is a prime.

Let N be the set of all positive integers. For any positive integer n, let S(n) denote the Smarandache function of n (see[1]). It is an obvious fact that $S(n) \leq n$. In this paper we consider the diophantine equation

(1) $$S(n) = n, \quad n \in N.$$

We prove a general result as follows:

Theorem. The equation (1) has only the solutions $n = 1,4$ or p, where p is a prime.

Proof. If $n = 1,4$ or p, then (1) holds. Let n be an another solution of (1). Then n must be a composite integer with $n > 4$. Since n is a composite integer, we have $n = uv$, where u,v are integers satisfying $u \geq v \geq 2$. If $u \neq v$, then we get $n \mid u!$. It implies that $S(n) \leq u = n / v < n$, a contradiction.

If $u = v$, then we have $n = u^2$ and $n | (2u)!$
It implies that $S(n) \leq 2u$. Since $n > 4$, we get $u > 2$ and $S(n) \leq 2u < u^2 = n$, a contradiction. Thus, (1) has only the solution $n = 1,4$ or p. The theorem is proved.

Reference

1. F Smarandache, A function in the number theory, Smarandache function J. 1 (1990), No.1, 3 - 17.

ON SMARANDACHE DIVISOR PRODUCTS

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. In this paper we give a formula for Smarandache divisor products.

Let n be a positive integer. In [1, Notion 20], the product of all positive divisors of n is called the Smarandache divisor product of n and denoted by $P_d(n)$. In this paper we give a formula of $P_d(n)$ as follows:

Theorem. Let $n = p_1^{r_1} \dots p_k^{r_k}$ be the factorization of n, and let

$$(1) \quad r(n) = \begin{cases} \frac{1}{2}(r_1+1)\dots(r_k+1), & \text{if n is not a square,} \\ \frac{1}{2}((r_1+1)\dots(r_k+1)-1), & \text{if n is a square.} \end{cases}$$

Then we have $P_d(n) = n^{r(n)}$.

Proof. Let f(n) denote the number of distinct positive divisors of n. It is a well known fact that

$$(2) \qquad f(n) = (r_1+1)\dots(r_k+1),$$

(See [2, Theorem 273]). If n is not a square and d is a positive divisor of n, then n/d is also a positive divisor of n with $n / d \neq d$. It implies that

$$(3) \qquad P_d(n) = n^{f(n)/2}.$$

Hence, by (1), (2) and (3), we get $P_d(n) = n^{r(n)}$.

If n is a square and d is a positive divisor of n with $d \neq \sqrt{n}$, then n/d is also a positive divisor of n with $n / d \neq d$. So we have

$$(4) \qquad P_d(n) = \frac{n^{f(n)/2}}{\sqrt{n}} = n^{(f(n-1)/2}.$$

Therefore, by (1), (2) and (4), we get $P_d(n) = n^{r(n)}$ too. The theorem is proved.

References.

1. Dumitrescu and V. Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
2. G.H.Hardy and e.M.Wright, An Introduction to the Theory of numbers, Oxford Univ. Press, Oxford, 1938.

ON THE SMARANDACHE N-ARY SIEVE

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let n be a positive integer with $n > 1$. In this paper we prove that the remaining sequence of Smarandache n-ary sieve contains infinitely many composite numbers.

Let n be a positive integer with $n > 1$. Let S_n denote the sequence of Smarandache n-ary sieve (see [1, Notions 29-31]). For example:

$S_2=\{1,3,5,9,11,13,17,21,25,27, \ldots\}$,

$S_3=\{1,2,4,5,7,8,10,11,14,16,17,19,20, \ldots\}$

In [1], Dumitrescu and Seleacu conjectured that S_n contains infinitely many composite numbers. In this paper we verify the above conjecture as follows:

Theorem. For any positive integer n with $n>1$, S_n contains infinitely many composite numbers.

Proof. By the definition of Smarandache n-ary sieve

(see [1, Notions 29-31]), the sequence S_n contains the numbers n^k+1 for any positive integer k. If k is an odd integer with $k > 1$, then we have

(1) $$n^k+1=(n+1)(n^{k-1}-n^{k-2}+\ldots+1).$$

We see from (1) that $(n+1) \mid (n^k+1)$ and n^k+1 is a composite number. Notice that there exist infinitely many odd integers k with $k>1$. Thus, S_n contains infinitely many composite numbers n^k+1. The theorem is proved.

References.

1. Dumitrescu and V. Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.

PERFECT POWERS IN THE SMARANDACHE PERMUTATION SEQUENCE

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. In this paper we prove that the Smarandache permutation sequience does not contain perfect powers.

Let $S = \{ Sn\}_{n=1}^{\infty}$ be the Smarandache permutation sequence. Then we have

(1) $s_1 = 12$, $s_2 = 1342$, $s_3 = 135642$, $s_4 = 13578642$, ...

In [1, Notion 6], Dumitrescu and Seleacu posed the following quiestion:

Question . Is there any perfect power belonging to S?

In this respect, Smarandache [2] conjectured: no! In this paper we verify the above conjecture as follows:

Theorem. The sequence S does not contain powers.

Proof. Let s_n be a perfect power. Since $2 \mid s_n$ by (1), then we have

(2) $4 \mid s_n$.

Since $s_1 = 12$ is not a perfect power, we get $n > 1$. Then

from (1) we get

(3) $s_n = 10^2 a + 42,$

where a is a positive integer. Notice that $4 \mid 10^2$ and $4 \nmid 42$. We find from (3) that $4 \nmid s_n$, which contradicts (2). Thus, the theorem is proved.

References

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
2. F.Smarandache, Only Problems, not Solutions! Xiquan Pub. House, Phoenix, Chicago, 1990.

ON SMARANDACHE PSEUDO - POWERS OF THIRD KIND

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let m be a positive integer with $m > 1$. In this paper we prove that there exist infinitely many m^{th} perfect powers which are Smarandache pseudo - m^{th} powers of third kind.

Let m be a positive integer with $m > 1$. For a positive integer a, if some nontrivial permutation of the digits is an m^{th} power, then a is called a Smarandache pseudo - m^{th} power. There were many questions concerning the number of Smarandache pseudo - m^{th} powers (see [1, Notions 71, 74 and 77]). In general, Smarandache [2] posed the following

Conjecture. For any positive integer m with $m > 1$, there exist infinitely many m^{th} powers which are Smarandache pseudo-m^{th} powers of third kind.

In this paper we verify the above conjecture as follows.

Theorem. For any positive integer m with $m > 1$, there exist infinitely many m^{th} powers are Smarandache pseudo-m^{th} powers of third kind.

Proof. For any positive integer k, the positive integer is an m^{th} power. Notice that $0 \ldots 01$ is a nontrivial permutation of the digits of 10^{km} and 1 is also an m^{th} power. It implies that there exist infinitely many Smarandache pseudo - m^{th}

powers of third kind. The theorem is proved.

References

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
2. F.Smarandache, Only Problems, not Solutions! Xiquan Pub. House, Phoenix, Chicago, 1993

AN IMPROVEMENT ON THE SMARANDACHE DIVISIBILITY THEOREM

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let a, n be positive integers. In this paper we prove that $n \mid (a^n - a)[n/2]!$

For any positive integer a and n, Smarandache [3] proved that

(1) $\quad n \mid (a^n - a)(n-1)!$.

The above division relation is the Smarandache divisibility theorem (see [1, Notions 126]). In this paper we give an improvement on (1) as follows:

Theorem. For any positive integers a and n, we have

(2) $\quad n \mid (a^n - a)[n/2]!$,

where $[n/2]$ is the largest integer which does not exceed $n/2$.

Proof. The division relation (2) holds for $n \le 9$, we may assume that $n > 9$. By Fermat's theorem (see [2, Theorem 71]), if n is a prime, then we have

(3) $\quad n \mid (a^n - a)$,

for any a. We see from (3) that (2) is true.

If n is a composite number, then we have $n = pd$, where p, d are integers satisfying $p \geq q \geq 2$. Further, if $p \neq q$, then we have $n | p!$ It implies that $n | (n / q)!$ Since $q \geq 2$, we get

(4) $n \mid [n/2]!$

If $p = q$, Then $n = p^2$ and

(5) $n \mid (2p)!$

Since $n > 9$, we have $n \geq 4^2$, $p \geq 4$ and $2p \leq n/2$. Hence, we see from (5) that (4) is also true in this case. The combination of (3) and (4), the theorem is proved.

References

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
2. G.H.Hardy and E. M.Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, Oxforf, 1936.
3. F.Smarandache, Problemes avec et sans ... problemes!, Somipress, Fes, Morocco, 1983.

ON PRIMES IN THE SMARANDACHE PIERCED CHAIN

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let $C = \{c_n\}_{n=1}^{\infty}$ be the Smarandache pierced chain. In this paper we prove that if $n > 2$, then $c_n / 101$ is not a prime.

For any positive integer n, let

(1) $$c_n = 101 * \underbrace{100010001 \ldots 0001}_{n-1 \text{ times}}.$$

Then the sequence $C = \{c_n\}_{n=1}^{\infty}$ is colled the Smarandache percied chain (see[2, Notion 19]). In [3], Smarandache asked the following question:

Question. How many $c_n / 101$ are primes?

In this paper we give a complete anser as follows:

Theorem. If $n > 2$, then $c_n / 101$ is not a prime.

Proof. Let $\zeta_n = e^{2\pi\sqrt{-1}/n}$ be a primitive roof of unity with the degree n, and let

$$f_n(x) = \prod_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} (x - \zeta_n^{\ k}).$$

Then $f_n(x)$ is a polynomial with integer coefficients. Further, it is a well known fact that if $x > 2$, then $f_n(x) > 1$ (see [1]). This implies that if x is an integer with $x > 2$, then $f_n(x)$ is an integer with $f_n(x) > 1$. On the other hand, we have

(2) $$x^n - 1 = \prod_{d/n} f_d(x).$$

We see from (1) that if $n > 1$, then

(3) $$\frac{c_n}{101} = 1 + 10^4 + 10^8 + \dots + 10^{4(n-1)} = \frac{10^{4n} - 1}{10^4 - 1}.$$

By the above definition, we find from (2) and (3) that

$$\frac{c_n}{101} = \left(\prod_{d \mid 4n} f_d(10)\right) / \left(\prod_{d \mid n} f_d(10)\right).$$

Since $n > 2$, we get $2n > 4$ and $4n > 4$. It implies that both $2n$ and $4n$ are divisors of $4n$ but not of 4. Therefore, we get from (4) that

(5) $$\frac{c_n}{101} = f_{2n}(10)\ f_{4n}(10)t,$$

where t is not a positive integer. Notice that $f_{2n}(10) > 1$ and $f_{4n}(10) > 1$. We see from (5) that $c_n / 101$ is not a prime. The theorem is proved.

References

1. G.D.Birkhoff and H.S.Vandiver, On the integral divisors of $a^n - b^n$, Ann. Of Math. (2), 5 (1904), 173 - 180.
2. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
3. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

PRIMES IN THE SEQUENCES $\{n^n + 1\}_{n=1}^{\infty}$ and $\{n^n + 1\}_{n=1}^{\infty}$

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let n be a positive integer. In this paper we prove that (i) if $n > 2$, then $n^n - 1$ is not a prime; (ii) if $n > 2$ and $n^n + 1$ is a prime, then $n = 2^{2^r}$, where r is a positive integer.

Let n be a positive integer. In [1, Problem 17], Smarandache posed the following questions

Question A. How many primes belong to the sequence $\{n^n - 1\}_{n=1}^{\infty}$?

Question B. How many primes belong to the sequence $\{n^n + 1\}_{n=1}^{\infty}$?

In this paper we prove the following results:

Theorem 1. 3 is the only prime belonging to $\{n^n - 1\}_{n=1}^{\infty}$.

Theorem 2. If $n > 2$ and $n^n + 1$ is a prime, then we we have $n = 2^{2^r}$, where r is a positive integer.

Proof of Theorem 1. If $n = 2$, then $2^2 - 1 = 3$ is a prime. If $n > 2$, then we have

(1) $n^n - 1 = (n-1)(n^{n-1} + n^{n-2} + \ldots + n + 1)$

Since $n - 1 > 1$ and $(n^{n-1} + n^{n-2} + \ldots + n + 1)$ if $n > 2$, we see from (1) that $n^n - 1$ is not a prime. The theorem is proved.

Proof of Theorem 2. Let $n^n + 1$ be a prime with $n > 2$. Since $n^n + 1$ is an even integer greater than 2 if $2 \nmid n$, we get $2 \mid n$. Let $n = 2^s n_1$, where s, n_1 are positive integers with $2 \nmid n_1$. If $n_1 > 1$, then we have

(2) $$n^n + 1 = (n^{2^s})^{n_1} + 1 = (n^{2^s} + 1)(n^{2^s(n_1-1)} - n^{2^s(n_1-2)} + \ldots - n^{2^s} + 1).$$

It is not a prime. So we have $n_1 = 1$ and $n = 2^s$. It implies that

(3) $$n^n + 1 = 2^{s \cdot 2^s} + 1.$$

By the same method, we see from (3) that if $n^n + 1$ is a prime, then s must be a power of 2. Thus, we get $n = 2^{2^r}$. The Theorem is proved.

Reference

1. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

ON THE SMARANDACHE PRIME ADDITIVE COMPLEMENT SEQUENCE

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let k be an arbitrary large positive integer. In this paper we prove that the Smarandache prime additive complement sequences includes the decreasing sequence $k, k-1, \ldots, 1, 0$.

For any positive integer n, let $p(n)$ be the smallest prime which does not excess n. Further let $d(n) = p(n) - n$. Then the sequence $D = \{d(n)\}_{n=1}^{\infty}$ is called the Smarandache prime additive complement sequence. Smarandache asked that if it is possible to as large as we want but finite decreasing sequence $k, k-1, \ldots, 1, 0$ included in D? Moreover, he conjectured that the answer is negative (see [1, Notion 46]). Howevwer, we shall give a positive answer for Smarandache's questions. In this paper we prove the following result:

Theorem. For an arbitrary large positive integer k, D includes the decreasing sequence $k, k-1, \ldots, 1, 0$.

Proof. Let $n = (k+1)! + 1$. Since $2, 3, \ldots, k+1$ are proper divisors of $(k+1)!$, then all numbers $n+1, n+2, \ldots n+k$ are composite numbers. It implies that $d(n) \geq k$. Therefore,

D includes the decreasing sequence k, k-1, ... , 1 , 0 . The theorem is proved.

Reference

1. Dumitrescu and Seleacu, Some Notions and Questions
In Number Theory, Erhus Univ. Press, Glendale, 1994

AN INEQUALITY FOR THE SMARANDACHE FUNCTION

by

Mihaly Bencze
2212 Săcele, Str. Harmanului 6,
Jud. Braşov, Romania

Let $S(m) = \min\{k \mid k \in N : m \mid k!\}$ be the Smarandache Function. In this paper we prove the following

THEOREM: $S(\prod_{k=1}^{m} m_k) \le \sum_{k=1}^{m} S(m_k)$.

We prove by induction. For m=1 it's true.
Let m=2, then we prove $S(m_1 m_2) \le S(m_1) + S(m_2)$.
We have $m_2 \mid S(m_2)!$ and if $r \ge S(m_1)$ then
$S(m_1)! \mid r(r-1)\ldots(r-S(m_1)+1)$.
If $t \mid S(n_1)!$ then $t \mid r(r-1)\ldots(r-S(n_1)+1)$ so
$m_1 m_2 \mid S(m_2)!(S(m_2)+1)\ldots(S(m_2)+S(m_1)) = (S(m_1)+S(m_2))!$
From this it results $S(m_1 m_2) \le S(m_1)+S(m_2)$.
We suppose they are true for m, and we prove for m+1.

$$S(\prod_{k=1}^{m+1} m_k)=S(m_1\prod_{k=2}^{m+1} m_k) \le S(m_1)+S(\prod_{k=2}^{m+1} m_k) \le S(m_1+ \sum_{k=1}^{m+1} S(m_k)=\sum_{k=1}^{m+1} S(m_k).$$

REFERENCE:

[1] Smarandache, Florentin, "A function in the Number Theory", <Smarandache Function Journal> (1990), No. 1, pp. 3-17.

ON SMARANDACHE SIMPLE FUNCTIONS

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Absatract. Let p be a prime, and let k be a positive integer. In this paper we prove that the Smarandache simple functions $S_p(k)$ satisfies $p \mid S_p(k)$ and $k(p-1) < S_p(k) \le kp$.

For any prime p and any positive integer k, let $S_p(k)$ denote the smallest positive integer such that $p^k \mid S_p(k)!$. Then $S_p(k)$ is called the Smarandache simple function of p and k (see [1, Notion 121]). In this paper we prove the following result.

Theorem. For any p and k, we have $p \mid S_p(k)$ and

(1) $\quad k(p-1) < S_p(k) \le kp$.

Proof. Let $a = S_p(k)$. Then a is the smallest positive integer such that

(2) $\quad p^k \mid a!$.

If $p \nmid a$, then from (2) we get $p^k \mid (a-1)!$, a contradiction. So we have $p \mid a$.

Since $(kp)! = 1 \ldots p \ldots (2p) \ldots (kp)$, we get $p^k \mid (kp)!$. It implies that

(3) $a \leq k p$.

On the other hand, let $p^r \mid a!$, where r is a positive integer. It is a well known fact that

(4) $$r = \sum_{i=1}^{\infty} [a / p^i]$$

where $[a / p^i]$ is the greatest integer which does not exceed a / p^i. Since $[a / p^i] \leq a / p^i$ for any i, we see from (4) that

(5) $$r < \sum_{i=1}^{\infty} (a / p^i) = a / (p - 1)$$

Further, since $k \leq r$ by (2), we find from (5) that

(6) $a > k (p - 1)$.

The combination of (3) and (6) yields (1). The theorem is proved.

Reference

1. Editor of Problem Section, Math. Mag 61 (1988), No.3, 202.

ON SMARANDACHE SIMPLE CONTINUED FRACTIONS

Charles Ashbacher[1]

Charles Ashbacher Technologies, Box 294
119 Northwood Drive, Hiawatha, IA 52233, USA

and

Maohua Le[1]

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let $A=\{a_n\}_{n=1}^{\infty}$ be a Smarandache type sequence. In this paper we show that if A is a positive integer sequence, then the simple continued fraction $[a_1, a_2, \ldots]$ is convergent.

Let $A=\{a_n\}_{n=1}^{\infty}$ be a Smarandache type sequence. Then The simple continued frction

$$(1) \qquad a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots\cdots}}}$$

is called the Smarandache simple continued fraction associated A (See [1]). By the usually symbol (see [2, Notion 10.1]), the continued frction (1) can be written as $[a_1, a_2, a_3, \ldots]$. Recently, Castillo [1] posed the following guestion:

Question. Is the continued fraction (1) convergent? In particular, is the continued fraction [1, 12, 123, ...] convergent?

In this paper we give a positive answer as follows.

Theorem. If A is a positive integer sequence, then the

[1]Editor's Note (M.L.Perez): This article has been done by each of the above authors independently.

continued fraction (1) is convergent.

Proof. If A is a positive integer sequence, then (1) is a usually simple continued fraction and its quotient are positive integers. Therefore, by [2,Theorem165], it is convergent. The Theorem is proved.

On applying [2, Theorems 165 and 176], we get a further result immediately.

Theorem 2. If A is an infinite positive integer sequence, then (1) is equal to an irrational number α. Further, if A is not periodic, then α is not an algebraic number of degree two.

References

1. J.Castillo, Smarandache continued fractions, Smarandache Notions J., to appear. Vol.9,No.1-2,40-42,1998.
2. G.H.Hardy and E.M.Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, Oxford, 1938.

NOTES ON PRIMES SMARANDACHE PROGRESSIONS

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. In this note we discuss the primes in Smarandache progressions.

For any positive integer n, let p_n denote the n^{th} prime.

For the fixed coprime positive integers a,b, let $P(a,b)=\{ap_n+b\}_{n=1}^{\infty}$. Then P(a,b) is called a Smarandache progression.
In [1, Problem 17], Smarandache possed the following questions:

Questions. How many primes belonng to P(a,b)?

It would seen that the answers of Smarandache's question is different from pairs (a,b). We now give some observable examples as follows:

Example 1. If a,b are odd integers, then ap_n+b is an even integer for n>1. It implies that P(a,b)contains at most one prime. In particular, P(1,1) contains only the prime 3.

Exemple 2. Under the assumption of twin prime conjecture that there exist infinitely many primes p such that p+2 is also a prime, then the progression P(1,2) contains infinitely

many primes.

Example 3. Under the assumption of Germain prime conjecture that there exist infinitely many primes p such that 2p+1 is also a prime, then the progression P(2,1) contains infinitely many primes.

Reference

1. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

THE PRIMES p WITH lg (p) =1

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. In this paper we prove that if $p=\overline{a_k \ldots a_1 a_0}$ is a prime satisfying $p>10$ and $lg(p)=1$, then $a_k=\ldots=a_1=a_0=1$ and $k+1$ is a prime.

Let $n=\overline{a_k \ldots a_1 a_0}$ be a decimal integer. Then the number of distinct digits of n is called the length of Smarandache generalized period of n and denoted by lg(n) (see [1, Notion 114]). In this paper we prove the following result.

Theorem. If $p=\overline{a_k \ldots a_1 a_0}$ is a prime satisfying $p>10$ and $lg(p)=1$, then we have $a_k=\ldots=a_1=a_0=1$ and $k+1$ is a prime.

Proof. Since $lg(p)=1$, we have $a_k=\ldots=a_1=a_0$. Let $a_0=a$, where a is an integer with $0<a\leq 9$. Then we have $a|p$. Since p is a prime and $p>10$, we get $a=1$ and

$$(1) \qquad p=\overline{1 \ldots 1}1=10^k+\ldots+10+1=\frac{10^{k+1}-1}{10-1},$$

where k is a positive integer. Since $k+1>1$, if $k+1$ is not a prime, then $k+1$ has a prime factor q such that $(k+1)/q>1$.

Hence, we see from (1) that

$$p=\frac{10^{k+1}-1}{10-1}=\left(\frac{10^{q}-1}{10-1}\right)\left(\frac{10^{k+1}-1}{10^{q}-1}\right)=(10^{q-1}+\ldots+10+1)(10^{(k+1)/q-1}+\ldots+10^{q}+1).$$

It implies that p is not a prime, a contradiction. Thus, if p is a prime, then k+1 must be a prime. The theorem is proved.

Reference

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994

SOME SOLUTIONS OF THE SMARANDACHE PRIME EQUATION

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let k be a positive integer with $k>1$. In this paper we give some prime solutions $(x_1,x_2,\ldots,x_k,y)$ of the diophantine equation $y=2x_1x_2\ldots x_k+1$ with $2<x_1<x_2<\ldots<x_k<y$.

Let k be a positive integer with $k>1$. In [4, Problem 11], Smarandache conjectured that the equation

(1) $\quad y=2x_1x_2\ldots x_k+1,\ 2<x_1<x_2<\ldots<x_k$

has infinitely many prime solutions $(x_1,x_2,\ldots x_k,y)$ for any k. This is a very dificult problem. The equation (1) is call the Smarandache prime equation (see [3, Notion 123]), while the authors gave solutions of (1) as follows.

$k=2$, $(x_1,x_2,y)=(17,19,647)$;
$k=3$, $(x_1,x_2,x_3,y)=(3,5,19,571)$

For any positive integer n, let p_n be the n^{th} odd prime, and let $q_n=2p_1p_2\ldots p_n+1$. In this paper, by the calculating

result of [1] and [2], we give nine other solutions as follows.

$$(x_1, x_2, \ldots, x_k, y) = (p_1, p_2, \ldots, p_k, q_k)$$

where k=4,10,66,138,139,311,368,495,514.

References

1. J.P.Buhler, R.E.Crandall, M.A.Penk, Primes of The form n!±1 and 2*3*5... p±1, Math. Comp. 38 (1982), 639-643.
2. H.Dubner, factorial and primorial primes, J.Recreational Math. 19 (1987), 197-203.
3. Dumitrescu and Seleacu, Some notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994
4. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

SMARANDACHE SERIES CONVERGES

by Charles Ashbacher
Charles Ashbacher Technologies
Hiawatha, Iowa

The Smarandache Consecutive Series is defined by repeatedly concatenating the positive integers on the right side of the previous element.

1, 12, 123, 1234, . . . , 123456789, 12345678910, 1234567891011, . . .

The Smarandache Reverse Sequence is defined by repeatedly concatenating the positive integers on the left side of the previous element.

1, 21, 321, 4321, . . . , 987654321, 10987654321, 1110987654321, . . .

a) Consider the series formed by summing the inverses of the Smarandache Consecutive Series

1/1 + 1/12 + 1/123 + 1/1234 + . . .

It is a simple matter to prove that this series is convergent. Forming the series

1/1 + 1/10 + 1/100 + 1/1000 + . . .

where it is well-known that this series is convergent to the number 10/9. Furthermore, the elements of the two series matched in the following correspondence

1/1 <= 1/1, 1/12 <= 1/10, 1/123 <= 1/100, ...

Therefore,by the ratio test, the sum of the inverses of the Smarandache Consecutive Series is also convergent.

b) Consider the series formed by taking the ratios of the terms of the consecutive sequence over the reverse sequence.

1/1 + 12/21 + 123/321 + 1234/4321 + . . .

In this case, it is straightforward to show that the series is divergent.

Consider an arbitrary element of the sequence

$$e(n) = \frac{a_1 a_2 \ldots a_k}{a_k \ldots a_2 a_1}$$

where the digit ak = 9. Clearly, e(n) > 1/10, as the numerator and denominator of this ratio have the same number of digits. Since there are an infinite number of such terms, the series contains an infinite number of terms all greater than 1/10. This forces divergence.

A NOTE ON PRIMES IN THE SEQUENCE $\{a^n + b\}_{n=1}^{\infty}$

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let a,b be integers such that gcd(a,b)=1 and $a \neq -1$, 0 or 1. Let $U(a,b)=\{a^n + b\}_{n=1}^{\infty}$. In this note we discuss the primes in U(a,b).

Let a,b be integers such that gcd(a,b)=1 and $a \neq -1$, 0 or 1. Let $U(a,b)=\{a^n + b\}_{n=1}^{\infty}$. In [1, Problem 17], Smarandache posed the following questions:

Question. How many primes belong to U(a,b)?

It would seem that the answers of this questions is different from different pairs (a,b). We now give some observable examples as follows:

Example 1. If a,b are odd integers, then a^n+b is either an even integer or zero. It implies that U(a,b) contains at most one prime. In particular, U(3,-1) contains only the prime 2, U(3,1) does not contain any prime.

Example 2. If a>2 and b=-1, then we have

$$(1) \qquad a^n+b = a^n-1 = (a-1)(a^{n-1} + a^{n-2} + \ldots + 1).$$

We see from(1)that $a^n + b$ is not a prime if n>1. It implies that U(a,b) contains at most one prime. In particular, U(4,-1) contains only the prime 3, U(10,-1) does not contain any prime.

Example 3. Under the assumption of Mersenne prime conjecture that there exist infinitely many primes with the form $2^n - 1$, then the sequence U(2,-1)contains infinitely many primes.

Reference

1. F.Smarandache, Only Problems, not Solutions!, Xiquan Pub. House, Phoenix, Chicago, 1990.

THE PRIMES IN THE SMARANDACHE SYMMETRIC SEQUENCE

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let $S=\{s_n\}_{n=1}^{\infty}$ Be the Smarandache symmetric sequence. In this paper we prove that if n is an even integer and $n/2 \neq 1 \pmod 3$, then s_n is not a prime.

Let $S=\{s_n\}_{n=1}^{\infty}$ be the Smarandache symmetric sequence, where

(1) $s_1=1$, $s_2=11$, $s_3=121$, $s_4=1221$, $s_5=12321$, $s_6=123321$, $s_7=1234321$, $s_8=12344321$, ...

Smarandache asked how many primes are there among S? (See [1, Notions 3]). In this paper we prove the following result:

Theorem. If n is an iven integer and $n/2 \neq 1 \pmod 3$, then s_n is not a prime.

Proof. If n is an even integer, then n=2k, where k is a positive integer. We see from (1) that

(2) $s_n = \overline{12 \ldots kk \ldots 21}$

It implies that

$$(3) \qquad s_n = 1*10^{t_1} + 2*10^{t_2} + \ldots + k*10^{t_k} + k*10^{t_{k+1}} + \ldots + 2*10^{t_{2k-1}} + 1*10^{t_{2k}},$$

where $t_1, t_2, \ldots, t_{2k}$ are nonnegative integers. Since $10^t \equiv 1 \pmod 3$ for any nonnegative integer t, we get from (3) that

$$(4) \qquad s_n \equiv 1+2+\ldots+k+k+\ldots+2+1 \equiv k(k+1) \pmod 3.$$

If $k \not\equiv 1 \pmod 3$, then either $k \equiv 0 \pmod 3$ or $k \equiv 2 \pmod 3$.
In both cases, we have $k(k+1) \equiv 0 \pmod 3$ and $3 \mid s_n$ by (4).
Thus, s_n is not a prime. The theorem is proved.

Reference

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.

ON SMARANDACHE GENERAL CONTINUED FRACTIONS

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. Let $A=\{a_n\}_{n=1}^{\infty}$ and $B=\{b_n\}_{n=1}^{\infty}$ be two Smarandache type sequences. In this paper we prove that if $a_{n+1} \geq b_n > 0$ and $b_{n+1} \geq b_n$ for any positive integer n, the continued fraction

$$(2) \qquad a_1 + \frac{b_1}{a_2 +}\ \frac{b_2}{a_3 +}\ \cdots$$

is convergent.

Let $A=\{a_n\}_{n=1}^{\infty}$ and $B=\{b_n\}_{n=1}^{\infty}$ be two Smarandache type sequences. Then the continued fraction

$$(1) \qquad a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots}}}$$

is called a Smarandache general continued fraction associated with A and B (see [1]). By using Roger's symbol, the continued fraction (1) can be written as

$$(2) \qquad a_1 + \frac{b_1}{a_2 +}\ \frac{b_2}{a_3 +}\ \cdots .$$

Recently, Castillo [1] posed the following question:

Question. Is the continued fractions $1 + \frac{1}{12}\ \frac{21}{+123}\ \frac{321}{+1234+}\cdots$

convergent?

In this paper we prove a general result as follows.

Theorem. If $a_{n+1} \geq b_n > 0$ and $b_{n+1} \geq b_n$ for any positive integer n, then the continued fraction (2) is convergent.

Proof. It is a well known fact that (2) is equal to the simple continued fraction

$$(2) \qquad a_1 + \frac{1}{c_1 +} \; \frac{1}{c_2 + \dots},$$

where

$$(4) \qquad c_{2t-1} = \frac{b_2 b_4 \dots b_{2t-2}}{b_1 b_3 \dots b_{2t-1}} a_{2t},$$

$$c_{2t} = \frac{b_1 b_3 \dots b_{2t-1}}{b_2 b_4 \dots b_{2t}} a_{2t+1}, \qquad t = 1,2,\dots .$$

Since $a_{n+1} \geq b_n > 0$ and $b_{n+1} \geq b_n$ for any positive n, we see from (4) that $c_n \geq 1$ for any n. It implees that the simple continued fraction (3) is convergent. Thus, the Smarandache general continued fraction (2) is convergent too. The theorem is proved.

Reference

1. J.Castillo, Smarandache continued fractions, Smarandache Notions J.,Vol.9,No.1-2,40-42,1998.

THE LOWER BOUND FOR THE SMARANDACHE COUNTER C(0,n!)

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. In this paper we prove that if n is an integer with $n\geq 5$, then the Smarandache counter $C(0,n!)$ satisfies $C(0,n!) > (1-5^{-k})(n+1)/4-k$, where $k=\lfloor \log n / \log 5 \rfloor$.

Let a be an integer with $0\leq a\leq 9$. For any positive decimal integer m, the number of a in digits of m is called the Smarandache counter of m with a. It is denoted by $C(a,m)$ (see [1,Notion 132]). Let n be a positive integer. In this paper we give a lower bound for $C(0,n!)$ as follows:

Theorem. If $n\geq 5$, then we have

$$C(0,n!)>1/4(1-5^{-k})(n+1)-k, \tag{1}$$

where $k=\lfloor \log n / \log 5 \rfloor$

Proof. Let

$$n!=\overline{a_s a_{s-1} \dots a_1 a_0}. \tag{2}$$

If $n\geq 5$, then we have $10|n!$ and $a_0=0$. Further, let $2^u \| n!$ and $5^v \| n!$. By [2, Theorem 1*11*1], we get

(3) $u=\sum_{r=1}^{\infty}[n/2^r]$, $v=\sum_{r=1}^{\infty}[n/5^r]$.

We see from (3) that $u \geq v$. It implies that there exist continuous v zeros $a_0=a_1=\ldots=a_{v-1}=0$ in (2). So we have

(4) $C(0,n!)\geq v$.

Let $k=[\log n/\log 5]$. Since $[n/5^r]=0$ if $r>k$, we see from (3) that

(5) $v=\sum_{r=1}^{\infty}[n/5^r]$

Since $[n/5^r] \geq n/5^r-(5^r-1)/5^r$, we get from (5)that

(6) $v \geq \sum_{r=1}^{k}(n/5^r-(5^r-1)/5^r)= 1/4(1-5^{-k})(n+1)-k$

Substitute (6) into (4) yelds (1) . The theorem is proved.

Reference

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
2. L.-K.Hua, Introduction to Number Theory, Springer, Berlin, 1982.

ON SMARANDACHE PSEUDO-PRIMES OF SECOND KIND

Maohua Le

Department of Mathematics, Zhanjiang Normal College
Zhanjiang, Guangdong, P.R.China.

Abstract. In this paper we prove that there exist infinitely many Smarandache pseudo-primes of second kind.

Let n be a composite number. If some permutation of the digits of n is a prime, then n is called a Smarandache pseudo-prime of second kind (see[1,Notion 65]). In this paper we prove the following result:

Theorem. There exist infinitely many Smarandache pseudo-primes of second kind.

Proof. Let the sequence $P=\{100r+1\}_{r=1}^{\infty}$. By Dirichlet's theorem (see[2,Theorem 15]), P contains infinitely many primes. Let

$$(1) \qquad p=\overline{a_k \dots a_2 a_1 a_0}$$

be a prime belonging to P. Then we have $a_0=1$ and $a_1=0$. Further let

$$(2) \qquad n=\overline{a_k \dots a_2 a_0 a_1}$$

Then we have $10 \mid n$, since $a_1=0$. Therefore, n is a composite number. Moreover, by (1) and (2), some permutation of the digits of n is prime p. It implies that n is a Smarandache pseudo-prime of second kind. Thus, there exist infinitely many Smarandache pseudo-primes of second kind. The theorem is proved.

Reference

1. Dumitrescu and Seleacu, Some Notions and Questions In Number Theory, Erhus Univ. Press, Glendale, 1994.
2. G.H.Hardy and E.M.Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, Oxford, 1938.

SUMMARY OF THE FIRST INTERNATIONAL CONFERENCE ON SMARANDACHE TYPE NOTIONS IN NUMBER THEORY (UNIVERSITY OF CRAIOVA, AUGUST 21-24 1997)

by Henry Ibstedt

The *First International Conference on Smarandache Notions in Number Theory* was held in Craiova , Romania, 21- 22 August 1997. The Organizing Committee had spared no effort in preparing programme, lodging and conference facilities. The Conference was opened by the professor Constantin Dumitrescu[1], chairman of the Organizing Committee and the initiator of the conference and a leading personality in Number Theory research. He welcomed all participants. Unfortunately professor Dumitrescu's state of health did not permit him to actively lead the conference, although he delivered his first paper later in the day and was present during most sessions. He requested the author of these lines to chair the first day of the conference, a task for which I was elected to continue for the rest of the conference.

In view of the above it is appropriate that I express mine and the other participants gratitude to the organizers and in particular to the Dumitrescu family who assisted throughout with social and arrangements and the facilities required for the smooth running of the conference. I would like to pay special tribute to professor Dumitrescu's son Antoniu Dumitrescu who presented his father's second paper on his behalf.

Unfortunately not all those who intended to participate in the conference were able to come. Their contributions which were submitted in advance have been gratefully received and are included in these proceedings.

A pre-conference session was held with professor V. Seleacu the day before the conference. This was held in french with Mrs Dumitrescu as interpreter. Prof. Seleacu showed some interesting work being conducted by the research group at Craiova University. Mrs Dumitrescu also acted actively during the conference to bridge language difficulties.

Special thanks were expressed at the conference to Dr. F. Luca, USA, who helped during sessions when translation from the romanian language to english was needed. In this context thanks are also due to my wife Anne-Marie Rochard-Ibstedt who made my participation possible by helping me drive from Sweden to Paris and then across Europe to Craiova. She was also active during the conference in taking photos and distributing documents.

Although united through the international language of Mathematics it was not always possible to penetrate presentations in such detail that extended discussions could take place after each session. Informal contacts between participants proved important and opportunities for this was given during breaks and joint dinners.

In the concluding remarks the chairman thanked the organizers and in particular professor Dumitrescu for having very successfully organized this conference. It was noted that the presentations were not made as an end in itself but as sources for further thought and research in this particular area of Number Theory, n.b. the very large number of open problems and notions formulated by Florentin Smarandache. The hope was expressed that the conference had linked together researchers for continuing exchange of views with our modern means of communication such as electronic mail and high speed personal computers.

Professor Dumitrescu thanked the chairman for his work.

Paris 26 March 1998.

[1] 1949-1997, Obituary in Vol. 8 of the Smarandache Notions Journal.

PREAMBLE TO
THE FIRST INTERNATIONAL CONFERENCE ON SMARANDACHE TYPE NOTIONS IN NUMBER THEORY (UNIVERSITY OF CRAIOVA, AUGUST 21-24 1997)

by Henry Ibstedt

Ladies and gentlemen,

It is for me a great honour and a great pleasure to be here at this conference to present some of the thoughts I have given to a few of the ideas and research suggestions given by Florentin Smarandache. In both of my presentations we will look at some integer sequences defined by Smarandache. As part of my work on this I have prepared an inventory of Smarandache sequences, which is probably not complete, but nevertheless it contains 133 sequences. I welcome contributions to complete this inventory, in which an attempt is also made to classify the sequences according to certain main types.

Before giving my first presentation I would like to say a few words about what eventually brought me here.

When I was young my interest in Mathematics began when I saw the beauty of Euclidean geometry - the rigor of a mathematical structure built on a few axioms which seemed the only ones that could exist. That was long before I heard of the Russian mathematician Lobachevsky and hyperbolic geometry. But my facination for Mathematics and numbers was awoken and who can dispute the incredible beauty of a formula like

$$e^{i\pi}+1=0$$

and many others. But there was also the disturbing fact that many important truths can not be expressed in closed formulas and that more often than not we have to resort to approximations and descriptions. For a long time I was fascinated by classical mechanics. Newton's laws provided an ideal framework for a great number of interesting problems. But Einstein's theory of relativity and Heisenberg's uncertainty relation put a stop to living and thinking in such a narrow world. Eventually I ended up doing computer applications in Atomic Physics. But also my geographical world became too narrow and I started working in developing countries in Africa, the far East and the Caribbean, far away from computers, libraries and contact with current research. This is when I returned to numbers and Number Theory. In 1979, when micro computers had just started making an impact, I bought one and brought it with me to the depths of Africa. Since then Computer Analysis in Number Theory has remained my major intellectual interest and stimulant.

With these words I would now like to proceed to the subject of this session.

The Smarandache Sequence Inventory

Compiled by Henry Ibstedt, July 1997

A large number of sequences which originate from F. Smarandache or are of similar nature appear scattered in various notes and papers. This is an attempt bring this together and make some notes on the state of the art of work done on these sequences. The inventory is most certainly not exhaustive. The sequences have been identified in the following sources where Doc. No. refers the list of Smarandache Documents compiled by the author. Nearly all of the sequences listed below are also found in Doc. No. 7: *Some Notions and Questions in Number Theory*, C. Dumitrescu and V. Seleacu, with, sometimes, more explicit definitions than those given below. Since this is also the most comprehensive list of Smarandache Sequences the paragraph number where each sequence is found in this document is included in a special column "D/S No"

Source	Seq. No.	Doc. No.
Numeralogy or Properties of Numbers	1-37	1
Proposed Problems, Numerical Sequences	38-46	2
A Set of Conjectures on Smarandache Sequences	47-57	16
Smarandache's Periodic Sequences	58-61	17
Only Problems, Not Solutions	62-118	4
Some Notions and Questions in Number Theory	119-133	7

Classification of sequences into eight different types (T):

The classification has been done according to what the author has found to be the dominant behaviour of the sequence in question. It is neither exclusive nor absolutely conclusive.

Recursive:	I	$t_n=f(t_{n-1})$, iterative, i.e. t_n is a function of t_{n-1} only.
	R	$t_n=f(t_i,t_j,\ldots)$, where $i,j<n$, $i\neq j$ and f is a function of at least two variables.
Non-Recursive:	F	$t_k=f(n)$, where f(n) may not be defined for all n, hence $k\leq n$.
Concatenation	C	Concatenation.
Elimination:	E	All numbers greater than a given number and with a certain property are eliminated.
Arrangement:	A	Sequence created by arranging numbers in a prescribed way.
Mixed operations:	M	Operations defined on one set (not necessarily N) to create another set.
Permutation:	P	Permutation applied on a set together with other formation rules.

Seq. No.	D/S No.	T	Name	Definition (intuitive and/or analytical)	State of the Art References
1		f	Reverse Sequence	1, 21, 321, 4321, 54321, ... 10987654321, ...	
2		R	Multiplicative Sequence	2, 3, 6, 12, 18, 24, 36, 48, 54, ... For arbitrary n_1 and n_2: n_k=Min(n_i·n_j), where k≥3 and j≤k, i≤k, i≠j.	
3		R	Wrong Numbers	n=$\underline{a_1a_2\ldots a_k}$, k≥2 (where $\underline{a_1a_2\ldots a_k}=a_1\cdot 10^{k-1}+a_2\cdot 10^{k-2}+\ldots+a_k$). For n>k the terms of the sequence a_1, a_2, ... a_n... are defined through $a_n = \prod_{i=n-k}^{n-1} a_i$. n is a wrong number if the sequence contains n.	Reformulated
4		f	Impotent Numbers	2, 3, 4, 5, 7, 9, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, ...A number n whose proper divisors product is less than n, i.e. {p, p^2; where p is prime}	
5		E	Random Sieve	1,5,6,7,11,13, 17, 19, 23, 25, ...General definition: Choose a positive number u_1 at random; -delete all multiples of all its divisors, except this number; choose another number u_2 greater than u_1 among those remaining; -delete all multiples of all its divisors, except this number; ... and so on.	
6		F	Cubic Base	0,1,2,3,4,5,6,7,10,11,12,13,14,15,16,17,20,21,22,23,24,25,26, ... Each number n is written in the cubic base.	
7		I	Anti-Symmetric Sequence	11,1212,123,123,12341234, ... ,12345678910111212345678910111 2, ...	
8		R	ss2(n)	1,2,5,26,29,677,680,701, ... ss2(n) is the smallest number, strictly greater than the previous one, which is the squares sum of two previous distinct terms of the sequence.	Ashbacher, C. Doc.14, p 25.
9		R	ss1(n)	1,1,2,4,5,6,16,17,18,20, ... ss1(n) is the smallest number, strictly greater than the previous one (for n≥3), which is the squares sum of one ore more previous distinct terms of the sequence.	
10		R	nss2(n)	1,2,3,4,6,7,8,9,11,12,14,15,16,18, ... nss2(n) is the smallest number, strictly greater than the previous one, which is NOT the squares sum of two previous distinct terms of the sequence.	Ashbacher, C. Doc.14, p 29.
11		R	nss1(n)	1,2,3,6,7,8,11,12,15,16,17,18,19, ... nss1(n) is the smallest number, strictly greater than the previous one, which is NOT the squares sum of one ore more previous distinct terms of the sequence.	
12		R	cs2(n)	1,2,9,730,737,389017001, 389017008,389017729, ... cs2(n) is the smallest number, strictly greater than the previous one, which is the cubes sum of two previous distinct terms of the sequence	Ashbacher, C. Doc.14, p 28.
13		R	cs1(n)	1,1,2,8,9,10,512,513,514,520, ... cs1(n) is the smallest number, strictly greater than the previous one (for n≥3), which is the cubes sum of one ore more previous distinct terms of the sequence.	
14		R	ncs2(n)	1,2,3,4,5,6,7,8,10,11,12,13,14,15, ... ncs2(n) is the smallest number, strictly greater than the previous one, which is NOT then cubes sum of two previous distinct terms of the sequence.	Ashbacher, C. Doc.14, p 32.
15		R	ncs1(n)	1,2,3,4,5,6,7,10,...,26,29, ... ncs1(n) is the smallest number, strictly greater than the previous one, which is NOT the cubes sum of one or more previous distinct terms of the sequence.	
16		R	SGR, General Recurrence Type Sequence	Let k≥j be natural numbers, and a_1, a_2, ..., a_k given elements, and R a j-relationsship (relation among j elements). Then: 1) The elements a_1, a_2, ..., a_k belong to SGR. 2) If m_1, m_2, ..., m_j belong to SGE, then R(m_1, m_2, ..., m_j) belongs to SGR too. 3) Only elements, obtained by rules 1) and/or 2) applied a finite number of times, belong to SGR.	
17		F	Non-Null Squares, ns(n)	1,1,1,2,2,2,2,3,4,4, ..., The number of ways in which n can be written as a sum of non-null squares. Example: $9=1^2+1^2+1^2+1^2+1^2+1^2+1^2+1^2+1^2=1^2+1^2+1^2+1^2+1^2+2^2=1^2+2^2+$	

				$2^2=3^2$. Hence ns(9)=4.	
18		F	Non-Null Cubes	1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,(8)3,(3)4, ...	
19		F	General Partition Sequence	Let f be an arithmetic function, and R a relation among numbers. {How many times can n be written in the form: $n=R(f(n_1),f(n_2), \ldots, f(n_k))$ for some k and $n_1, n_2, \ldots n_k$ such that $n_1+n_2+\ldots+n_k=n$?}.	
20		C	Concatenate Seq.	1,22,333,4444,55555,666666,....	
21		F	Triangular Base	1,2,10,11,12,100,101,102,110,1000, ... Numbers written in triangular base, defined as follows: $t_n=n(n+1)/2$ for $n\geq1$.	
22		F	Double Factorial Base	1,10,100,101,110,200,201,1000, ...	
23		R	Non-Multiplicative Sequence	Let m_1, m_2, ... m_k be the first k given terms of the sequence, where $k\geq2$; then m_i, for $i\geq k+1$, is the smallest number not equal to the product of the k previous terms.	
24		R	Non-Arithmetic Sequence	If m_1, m_2 are the first k two terms of the sequence, then m_k for $k\geq3$, is the smallest number such that no 3-term arithmetic progression is in the sequence.	Ibstedt, H. Doc. 19, p. 1.
25		R	Prime Product Sequence	2,7,31,211,2311,30031,510511, ... $p_n=1+p_1p_2\ldots p_n$, where p_k is the k-th prime.	Ibstedt, H. Doc. 19, p.4.
26		R	Square Product Sequence	2,5,37,577,14401,518401,25401601, ... $S_n=1+s_1s_2\ldots s_n$, where s_k is the k-th square number.	Ibstedt, H. Doc. 19, p. 7.
27		R	Cubic Product Sequence	2,9,217,13825,1728001,373248001, ... $C_n=1+c_1c_2\ldots c_n$, where c_k is the k-th cubic number.	
28		R	Factorial Product Sequence	1,3,13,289,34561,24883201, ... $F_n=1+f_1f_2\ldots f_n$, where f_k is the k-th factorial number.	
29		R	U-Product Sequence (Generalization)	Let u_n, $n\geq1$, be a positive integer sequence. Then we define a U-sequence as follows: $U_n=1+u_1u_2\ldots u_n$.	
30		R	Non-Geometric Sequence	1,2,3,5,6,7,8,10,11,13,14,15, ... Definition: Let m_1 and m_2 be the first two term of the sequence, then m_k ,for $k\geq3$, is the smallest number such that no 3-term geometric progression is in the sequence.	
31		F	Unary Sequence	11, 111, 11111, 1111111, 11111111111, ... $u_n=11\ldots1$, p_n digits of "1", where p_n is the n-th prime.	
32		F	No Prime Digits Sequence	1,4,6,8,9,10,11,1,1,14,1,16,1,18, ... Take out all prime digits from n.	
33		F	No Square Digits Sequence	2,3,4,6,7,8,2,3,5,6,7,8,2,2,22,23,2,25, ... Take out all square digits from n.	
34		C	Concatenated Prime Sequence	2,23,235,2357, 235711, 23571113, ...	Ibstedt, H. Doc. 19, p. 13.
35		C	Concatenated Odd Sequence	1,13,135,1357,13579,1357911,135791113, ...	Ibstedt, H. Doc. 19, p. 12
36		C	Concatenated Even Sequence	2,24,246,2468,246810,24681012, ...	Ibstedt, H. Doc. 19, p. 12.
37		C	Concatenated S-Sequence (Generalization)	Let s_1, s_2, s_3, ... s_n be an infinite integer sequence. Then s_1, s_1s_2, $s_1s_2s_3$, $s_1s_2s_3\ldots s_n$ is called the concatenated S-sequence.	
38		A	Crescendo Sub-Seq.	1, 1,2 1,2,3 1,2,3,4 1,2,3,4,5 ...	
39		A	Decrescendo Sub-S.	1, 2,1 3,2,1 4,3,2,1 5,4,3,2,1 ...	
40		A	Cresc. Pyramidal Sub-S	1, 1,2,1 1,2,3,2,1, 1,2,3,4,3,2,1 ...	
41		A	Decresc. Pyramidal Sub-S	1, 2,1,2, 3,2,1,2,3, 4,3,2,1,2,3,4, ...	
42		A	Cresc. Symmetric Sub-S	1, 1, 2,1,1,2, 3,2,1,1,2,3, 1,2,3,4,4,3,2,1 ...	
43		A	Decresc. Symmetric Sub-S	1,1, 2,1,1,2, 3,2,1,1,2,3, 4,3,2,1,1,2,3,4, ...	
44		A	Permutation Sub-S	1, 2, 1,3,4,2, 1,3,5,6,4,2, 1,3,5,7,8,6,4,2,1, ...	
45		E	Square-Digital Sub-Sequence	0, 1, 4, 9, 49, 100, 144, 400, 441, ...	Ashbacher, C. Doc.14, p 45.
46		E	Cube-Digital Sub-Sequence	0, 1, 8, 1000, 8000, ...	Ashbacher, C. Doc.14, p 46.
47		E	Prime-Digital Sub-Sequence	2, 3, 5, 7, 23,37,53,73	Ashbacher, C. Doc.14, p 48. Ibstedt, H. Doc. 19, p. 9.

48		E	Square-Partial-Digital Sub-Seq. .	49, 100, 144, 169, 361, 400, 441, ...Squares which can be partitioned into groups of digits which are perfect squares	Ashbacher, C. Doc.14, p 44.
49		E	Cube-Partial-Digital Sub-Sequence	1000, 8000, 10648, 27000, ...	Ashbacher, C. Doc.14, p 47.
50		E	Prime-Partial-Digital Sub-Sequence	23, 37, 53, 73, 113, 137, 173, 193, 197, ... Primes which can be partitioned into groups of digits which are also primes.	Ashbacher, C. Doc.14, p 49.
51		F	Lucas-Partial Digital Sub-Sequence	123, ... (1+2=3, where 1,2 and 3 are Lucas numbers)	Ashbacher, C. Doc.14, p 34.
52		E	f-Digital Sub-Sequence	If a sequence $\{a_n\}$, $n\geq 1$ is defined by $a_n=f(n)$ (a function of n), then the f-digital subsequence is obtained by screening the sequence and selecting only those terms which can be partitioned into two groups of digits g_1 and $g_2=f(g_1)$.	
53		E	Even-Digital Sub-S.	12, 24, 36, 48, 510, 612, 714, 816, 918, 1020, 1122, 1224, ...	Ashbacher, C. Doc.14, p 43.
54		E	Lucy-Digital Sub-S.	37, 49, ... (i.e. 37 can be partioned as 3 and 7, and $l_3=7$; the lucky numbers are 1,3,7,9,113,15,21,25,31,33,37,43,49,51,63, ...	Ashbacher, C. Doc.14, p 51.
55		M	Uniform Sequence	Let n be an integer$\neq 0$, and d_1, d_2, ... d_r distinct digits in base B. Then: multiples of n, written with digits d_1, d_2, ... d_r only (but all r of them), in base B, increasingly ordered, are called the uniform S.	
56		M	Operation Sequence	Let E be an ordered set of elements, $E=\{e_1, e_2, ...\}$ and θ a set of binary operations well defined for these elements. Then: $a_1\varepsilon\{e_1, e_2, ...\}$, $a_{n+1}=\min\{e_1\ \theta_1\ e_2\ \theta_2 \ldots \theta_A\ e_{n+1}\}>a_n$ for $n\geq 1$.	
57		M	Random Operation Sequence	Let E be an ordered set of elements, $E=\{e_1, e_2, ...\}$ and θ a set of binary operations well defined for these elements. Then: $a_1\varepsilon\{e_1, e_2, ...\}$, $a_{n+1}=\{e_1\ \theta_1\ e_2\ \theta_2 \ldots \theta_A\ e_{n+1}\}>a_n$ for $n\geq 1$.	
58		M	N-digit Periodic Sequence	42,18,63,27,45,09,81,63,27, ... Start with a positive integer N with not all its digits the same, and let N' be its digital reverse. Put $N_1=\lvert N-N'\rvert$ and let N_1' be the digital reverse of N_1. Put $N_2=\lvert N_1-N_1'\rvert$, and so on.	Ibstedt, H. Doc. 20, p. 3.
59		M	Subtraction Periodic Sequence	52,24,41,13,30,02,19,90,08,79,96,68,85,57,74,46,63,35,52, ... Let c be a fixed positive integer. Start with a positive integer N and let N' be its digital reverse. Put $N_1=\lvert N'-c\rvert$ and let N_1' be the digital reverse of N_1. Put $N_2=\lvert N_1'-c\rvert$, and so on.	Ibstedt, H. Doc. 20, p. 4.
60		M	Multiplication Periodic Sequence	68,26,42,84,68, ... Let c>1 be a fixed integer. Start with a positive integer N, multiply each digit x of N by c and replace that digit by the last digit of cx to give N_1, and so on.	Ibstedt, H. Doc. 20, p. 7.
61		M	Mixed Composition Periodic Sequence	75,32,51,64,12,31,42,62,84,34,71,86,52,73,14,53,82,16,75, ... Let N be a two-digit number. Add the digits, and add them again if the sum is greater then 10. Also take the absolute value of their difference. These are the first and second digits of N_1. Now repeat this.	Ibstedt, H. Doc. 20, p. 8.
62	1	I	Consecutive Seq.	1, 12, 123, 12345, 123456, 1234567,	
63	2	I/P	Circular Sequence	1, (12, 21), (123, 231, 312), (1234, 2341, 3412, 4123), ...	Kashihara, K. Doc. 15, p. 25.
64	3	A	Symmetric Sequence	1, 11, 121, 1221, 12321, 123321, 1234321, 12344321,	Ashbacher, C. Doc.14, p 57.
65	4	A	Deconstructive Sequence	1, 23, 456, 7891, 23456, 789123, 4567891, 23456789, 123456789, 1234567891,	Kashihara, K. Doc. 15, p.6.
66	5	A	Mirror Sequence	1, 212, 32123, 4321234, 543212345, 65432123456,	Ashbacher, C. Doc.14, p 59.
67	6 7	A/P	Permutation Sequence Gen. in doc. no. 7	12, 1342, 135642, 13578642, 13579108642, 135791112108642,1357911131412108642, ..	Ashbacher, C. Doc.14, p 5.
68 *		M	Digital Sum	(0,1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,8,9,10), (2,3,4,5,6,7,8,9,10,11), .. ($d_s(n)$ is the sum of digits)	Kashihara, K. Doc. 15, p.6.
69 *		M	Digital Products	0,1,2,3,4,5,6,7,8,9,0,1,2,3,4,5,6,7,8,9,0,2,4,6,8,19,12,14,16,18, 0,3,6,9,12,15,18,21,24,27,0,4,8,12,16,20,24,28,32,36,0,5,10,1 5,20,25, ... $d_p(n)$ is the product of digits	Kashihara, K. Doc. 15, p.7.
70	15	F	Simple Numbers	2,3,4,5,6,7,8,9,10,11,13,14,15,17, .. A number is called a	Ashbacher, C.

				simple number if the product of its proper divisors is less than or equal to n.	Doc.14, p20.
71	19	I	Pierced Chain	101, 1010101, 10101010101, 101010101010101, ... c(2)=101*10001, c(3)=101*100010001, etc Qn. How many c(n)/100 are primes?	Ashbacher, C. Doc.14, p 60. Kashihara, K. Doc. 15, p. 7.
72	20	F	Divisor Products	1,2,3,8,5,36,7,64,27,100,11,1728,13,196,225,1024,17,... $p_d(n)$ is the product of all positive divisors of n.	Kashihara, K. Doc. 15, p. 8.
73	21	F	Proper Divisor Products	1,1,1,2,1,6,1,8,3,10,1,144,1,14,15,64,1.324, ... $p_d(n)$ is the product of all positive divisors of n except n.	Kashihara, K. Doc. 15, p. 9.
74	22	F	Square Complements	1,2,3,1,5,6,7,2,1,10,11,3,14,15,1,17,2,19,5,21,22,23,6,1,26,... For each integer n find the smallest integer k such that nk is a perfect square.	Ashbacher, C. Doc.14, p 9. Kashihara, K. Doc. 15, p. 10.
75	23 24	F	Cubic Complements Gen. to m-power complements in doc. no. 7	1,4,9,2,25,36,49,1,3,100,121,18,169,196,225,4,289, ... For each integer n find the smallest integer k such that nk is a perfect cube.	Ashbacher, C. Doc.14, p 9. Kashihara, K. Doc. 15, p. 11.
76	25 26	E	Cube free sieve Gen. in doc. no. 7	2,3,4,5,6,7,9,10,11,12,13,14,15,17,18,19,20,21,22,23,24,25,26 ,28, ...	
77	27	E	Irrational Root Sieve	2.3.5.6.7.10.11.12.13.14.15.17. Eliminate all a^k, when a is squarefree.	
78	37	F	Prime Part (Inferior)	2,3,3,5,5,7,7,7,7,11,11,13,13,13,13,17,17,19,19,19,19,23,23,2 3,23,23,23, ... For any positive real number n $p_p(n)$ equals the largest prime less than or equal to n.	Kashihara, K. Doc. 15, p. 12.
79	38	F	Prime Part (Superior)	2,2,2,3,5,5,7,7,11,11,11,11,13,13,17,17,17,17,19,19,23,23,23, 23, ... For any positive real number n $p_p(n)$ equals the smallest prime number greater than or equal to n.	Kashihara, K. Doc. 15, p. 12.
80	39	F	Square Part (Inferior)	0,1,1,1,4,4,4,4,4,9,9,9,9,9,9,9, ... The largest square less than or equal to n.	Kashihara, K. Doc. 15, p. 13.
81	40	F	Square Part (Superior)	0,1,4,4,4,9,9,9,9,9, ... The smallest square greater than or equal to n.	Kashihara, K. Doc. 15, p. 13.
82	41	F	Cube Part (Inferior)	0,1,1,1,1,1,1,1,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8, ... The largest cube less than or equal to n.	
83	42	F	Cube Part (Superior)	0,1,8,8,8,8,8,8,8, ... The smalest cube greater than or equal to n.	
84	43	F	Factorial Part (Inferior)	1,2,2,2,2, (18)6, ... $F_p(n)$ is the largest factorial less than or equal to n.	
85	44	F	Factorial Part (Superior)	1,2, (4)6, (18)24, (11)120, ... $f_p(n)$ is the smallest factorial greater than or equal to n.	
86	45	F	Double Factorial Complements	1,1,1,2,3,8,15,1,105,192,945,4,10395,46080,1,3,2027025, ... For each n find the smallest k such that nk is a double factorial, i.e. nk=1·3·5·7·9·...·n (for odd n) and nk=2·4·6·8·...·n (for even n)	
87	46	F	Prime additive complements	1,0,0,1,0,1,0,3,2,1,0,1,0,3,3,2, ... t_n=n+k where k is the smallest integer for which n+k is prime (reformulated).	Ashbacher, C. Doc.14, p 21. Kashihara, K. Doc. 15, p. 14.
88		F	Factorial Quotients	1,1,2,6,24,1,720,3,80,12,3628800, ... t_n=nk where k is the smallest integer such that nk is a factorial number (reformulated).	Kashihara, K. Doc. 15, p. 16.
89 *		F	Double Factorial Numbers	1,2,3,4,5,6,7,4,9,10,11,6,... $d_f(n)$ is the smallest integer such that $d_{f(n)}$!! is a multiple of n.	
90	55	F	Primitive Numbers (of power 2)	2,4,4,6,8,8,8,10,12,12,14,16,16,16,16, ... $S_2(n)$ is the smallest integer such that $S_2(n)$! is divisible by 2^n.	Important
91	56 57	F	Primitive Numbers (of power 3) Gen. to power p, p prime.	3,6,9,9,12,15,18,18, ... $S_3(n)$ is the smallest integer such that $S_3(n)$! is divisible by 3^n.	Kashihara, K. Doc. 15, p. 16.
92		M	Sequence of Position	Definition: Unsolved problem: 55	
93	58	F	Square Residues	1,2,3,2,5,6,7,2,3,10,11,6, ... $S_r(n)$ is the largest square free number which divides n.	
94	59 60	F	Cubical Residues Gen. to m-power residues.	1,2,3,4,5,6,7,9,10,11,12,13, ... $C_r(n)$ is the largest cube free number which divides n.	
95	61	F	Exponents (of power	0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4, ... $e_2(n)$=k if 2^k divides n but	Ashbacher, C.

			2)	2^{k+1} if it does not.	Doc.14, p 22.
96	62 63	F	Exponents (of power 3). Gen. to exp. of power p	0,0,1,0,0,1,0,0,2,0,0,1,0,0,1,0,0,2,... $e_2(n)=k$ if 3^k divides n but 3^{k+1} if it does not.	Ashbacher Doc.14, p 24.
97	64 65 66	F/P	Pseudo-Primes of first kind. Ext. to second and third kind in doc. no. 7.	2,3,5,7,11,13,14,16,17,19,20, ... A number is a pseudo-prime if some permutation of its digits is a prime (including the identity permutation).	Kashihara, K. Doc. 15, p. 17.
98	69 70 71	F/P	Pseudo-Squares of first kind. Ext. to second and third kind in doc. no. 7.	1,4,9,10,16,18,25,36,40, ... A number is a pseudo-square if some permutation of its digits is a perfect square (including the identity permutation).	Ashbacher, C. Doc.14, p 14. Kashihara, K. Doc. 15, p. 18.
99	72 73 74 75 76 77	F/P	Pseudo-Cubes of first kind. Ext. to second and third kind in doc. no. 7. (Gen. Pseudo-m-powers)	1,8,10,27,46,64,72,80,100, ... A number is a pseudo-cube if some permutation of its digits is a cube (including the identity permutation).	Ashbacher, C. Doc.14, p 14. Kashihara, K. Doc. 15, p. 18.
100	78 79 80	F/P	Pseudo-Factorials of first kind. Ext. to second and third kind in doc. no. 7.	1,2,6,10,20,24,42,60,100,102,120, ... A number is a pseudo-factorial if some permutation of its digits is a factorial number (including the identity permutation).	
101	81 82 83	F/P	Pseudo-Divisors of first kind. Ext. to second and third kind in doc. no. 7.	1,10,100,1,2,10,20,100,200,1,3,10,30, ... A number is a pseudo-divisor of n if some permutation of its digits is a divisor of n (including the identity permutation).	
102	84 85 86	F/P	Pseudo-Odd Numbers of first kind. Ext. to second and third kind in doc. no. 7.	1,3,5,7,9,10,11,12,13,14,15,16,17, ... A number is a pseudo-odd number if some permutation of its digits is an odd number.	Ashbacher, C. Doc.14, p 16.
103	87	F/P	Pseudo-Triangular Numbers	1,3,6,10,12,15,19,21,28,30,36, ... A number is a pseudo-triangular number if some permutation of its digits is a triangular number.	
104	88 89 90	F/P	Pseudo-Even Numbers of first kind. Ext. to second and third kind in doc. no. 7.	0,2,4,6,8,10,12,14,16,18,20,21,22,23, ... A number is a pseudo-even number if some permutation of its digits is an even number.	Ashbacher, C. Doc.14, p17.
105	91 92 93 94 95 96	F/P	Pseudo-Multiples (of 5) of first kind. Ext. to second and third kind in doc. no. 7. (Gen. to Pseudo-multiples of p.)	0,5,10,15,20,25,30,35,40,45,50,51, ... A number is a pseudo-multiple of 5 if some permutation of its digits is a multiple of 5 (including the identity permutation).	Ashbacher, C. Doc.14, p19.
106	100	F	Square Roots	0,1,1,1,2,2,2,2,2,3,3,3,3,3,3,3, ... $s_q(n)$ is the superior integer part of the square root of n.	
107	101 102	F	Cubical Roots Gen. to m-power roots $m_c(n)$	0,1,1,1,1,1,1,1, 19(2), 37(3), ... $c_q(n)$ is the superior integer part of the cubical root of n.	
108	47	F	Prime Base	0,1,10,100,101,1000,1001,10000,10001,10010, ... See Unsolved problem: 90	Kashihara, K. Doc. 15, p. 32.
109	48 49	F	Square Base Gen. to m-power base and gen. base (Unsolved problem 93)	0,1,2,3,10,11,12,13,20,100,101, ... See Unsolved problem: 91	
110	28	M	Odd Sieve	7,13,19,23,25,31,33,37,43, ... All odd numbers that are not equal to the difference between two primes.	
111	29	E	Binary Sieve	1,3,5,9,11,13,17,21,25, ... Starting to count on the natural numbers set at any step from 1: -delete every 2-nd numbers; -delete, from the remaining ones, every 4-th numbers ... and so on: delete, from the remaining ones, every 2^k-th numbers, k=1,2,3,... .	Ashbacher, C. Doc.14, p 53.
112	30 31	E	Trinary Sieve Gen. to n-ary sieve	1,2,4,5,7,8,10,11,14,16,17, ... (Definition equiv. to 114)	Ashbacher, C. Doc.14, p 54.
113	32	E	Consecutive Sieve	1,3,5,9,11,17,21,29,33,41,47,57, ... From the natural numbers: - keep the first number, delete one number out	Ashbacher, C. Doc.14, p 55.

				of 2 from all remaining numbers; - keep the first remaining number, delete one number out of 3 from the next remaining numbers; and so on	
114	33	E	General-Sequence Sieve	Let $u_i>1$, for i=1, 2, 3, … , be a strictly increasing integer sequence. Then: From the natural numbers: -keep one number among 1,2,3, … ,u_1-1 and delete every u_1-th numbers; -keep one number among the next u_2-1 remaining numbers and delete every u_2-th numbers; and so on, for step k (k≥1): keep one number among the next u_k-1 remaining numbers and delete every u_k-th numbers;	
115	36	M	General Residual Sequence	$(x+C_1)$ …. $(x+C_{F(m)})$,m=2, 3, 4, … , where C_i, $1\leq i\leq F(m)$, forms a reduced set of residues mod m. x is an integer and f is Euler's totient.	Kashihara, K. Doc. 15, p. 11.
116		M	Table:(Unsolved 103)	6,10,14,18,26,30,38,42,42,54,62,74,74,90, … t_n is the largest even number such that any other even number not exceeding it is the sum of two of the first n odd primes.	Kashihara, K. Doc. 15, p. 19.
117		M	Second Table	9,15,21,29,39,47,57,65,71,93,99,115,129,137, … v_n is the largest odd number such that any odd number ≥9 not exceeding it is the sum of three of the first n odd primes.	Kashihara, K. Doc. 15, p. 20.
118		M	Second Table Sequence	0,0,0,0,1,2,4,4,6,7,9,10,11,15,17,16,19,19,23, … a_{2k+1} represents the number of different combinations such that 2k+1 is written as a sum of three odd primes.	Kashihara, K. Doc. 15, p. 20.
119	34	E	More General-Sequence Sieve	Let $u_i>1$, for i=1, 2, 3, … , be a strictly increasing integer sequence, and $v_i\leq u_i$ another positive integer sequence. Then: From the natural numbers: -keep the v_1-th number among 1,2,3, … ,u_1-1 and delete every u_1-th numbers; -keep the v_2-th number among the next u_2-1 remaining numbers and delete every u_2-th numbers; and so on, for step k (k≥1): -keep the v_k-th number among the next u_k-1 remaining numbers and delete every u_k-th numbers;	
120	35	F	Digital Sequences Special case: Construction sequences	In any number base B, for any given infinite integer or rational sequence s_1, s_2, s_3, …, and any digit D from 0 to B-1, build up a new integer sequence which associates to s_1 the number of digits of D of s_1 in base B, to s_2 the number of digits D of s_2 in base B, and so on.	
121	50	F	Factorial Base	0,1,10,11,20,21,100,101,110,111,120,121,200,201,210,211, … (Each number n written in the Smarandache factorial base.)(Smarandache defined over the set of natural numbers the following infinite base: for k≥1, f_k=k!)	
122	51	F	Generalized Base	(Each number n written in the Smarandache generalized base.)(Smarandache defined over the set of natural numbers the following infinite base: $1=g_0<g_1<\ldots<g_k<\ldots$)	
123	52	F	Smarandache Numbers	1,2,3,4,5,3,7,4,6,5, … . S(n) is the smallest integer such that s(n)! is divisible by n.	
124	53	F	Smarandache Quotients	1,1,2,6,24,1,720,3,80,12,3628800, … . For each n find the smallest k such that nk is a factorial number.	
125	54	F	Double Factorial Numbers	1,2,3,4,5,6,7,4,9,10,11,6,13, … . $d_f(n)$ is the smallest integer such that $d_f(n)$!! is a multiple of n.	
126	67	R	Smarandache almost Primes of the first kind	$a_1\geq 2$, for n≥2 a_n= the smallest number that is not divisible by any of the previous terms.	
127	68	R	Smarandache almost Primes of the second kind	$a_1\geq 2$, for n≥2 a_n= the smallest number that is coprime with all the previous terms.	
128	97	C R	Constructive Set S (of digits 1 and 2)	I: 1,2 belong to S II: if a and b belong to S, then ab (concatenation) belongs to S III: Only elements obtained be applying rules I and II a finite number of times belong to S	
129	98 99	C R	Constructive Set S (of digits 1,2 and 3) Gen. Constructive Set (of digits d_1, d_2, … d_m) 1≤m≤9.	I: 1,2 , 3 belong to S II: if a and b belong to S, then ab (concatenation) belongs to S III: Only elements obtained be applying rules I and II a finite number of times belong to S	
130	104	F	Goldbach-Smarandache Table	6,10,14,18,26,30,38,42,42,54, … .t(n) is the largest even number such that any other even number not exceeding it is the sum of two of the first n odd primes.	

131	105	F	Smarandache-Vinogradov Table	9,15,21,29,39,47,57,65,71,93, … . V(n) is the largest odd number such that any odd number ≥ 9 not exceeding it is the sum of three of the first n odd primes.	
132	106	F	Smarandache-Vinogradov Sequence	0,0,0,0,1,2,4,4,6,7,9,10, … .a(2k+1) represents the number of different combinations such that 2k+1 is written as a sum of three odd primes.	
133	115	F	Sequence of Position	Let $\{x_n\}$, $n \geq 1$, be a sequence of integers and $0 \leq k \leq 9$ a digit. The Smarandache sequence of position is defined as $U_n^{(k)} = U^{(k)}(x_n) = \max\{i\}$ if k is the 10-th digit of x_n else -1.	

List of Smarandache Documents

Compiled by Henry Ibstedt, July 1997

Doc. No	Title	Author/Ref.	Year	ISBN nr
1	Numeralogy or Properties of Numbers	Smarandache, F.	1975	Univ. Craiova Archives
2	Proposed Problems, Numerical Sequences	Smarandache, F.	1975	Univ. Craiova
3	Smarandache Function Journal	Vol. 1	1992	?
4	Only Problems, Not Solutions (Edition?)	Smarandache, F.	1993	1-879585-00-6
5	Only Problems, Not Solutions, 4th ed.	Smarandache, F.	1993	1-879585-00-6
6	Smarandache Function Journal	Vol. 2-3, No. 1	1993	1053-4792
7	Some Notions and Questions in Number Theory	Dumitrescu, C. Seleacu, V.	1994	1-879585-48-0
8	Smarandache Function Journal	Vol. 4-5, No. 1	1994	1053-4792
9	Smarandache Function Journal	Vol. 6, No. 1	1995	1053-4792
10	An Introduction to the Smarandache Function	Ashbacher, C.	1995	1-879585-49-9
11	Smarandache Notions Journal	Vol. 7, No. 1-2-3	1996	1084-2810
12	The Most Paradoxxist Mathematician of the World	Le Charles, T.	1996	1-879585-52-9
13	Collected Papers	Smarandache, F.	1996	973-9205-02-X
14	Collection of Problems on Smarandache Notions	Ashbacher, C.	1996	1-879585-50-2
15	Comments and Topics on Smarandche Notions and Problems	Kashihara, K.	1996	1-897585-55-3
16	A Set of Conjectures on Smarandache Sequences	Smith, Sylvester	1996	Bulletin of Pure and Applied Sciences
17	Smarandache's Periodic Sequences (Sequences of Numbers)	Popov, M.R. Smarandache, F.	1996	Mathematical Spectrum, Vol 29, No 1 (Univ. Craiova Conf. 1975)
18	Surfing on the Ocean of Numbers - a Few Smarandache Notions and Similar Topics	Ibstedt, H.	1997	1-879585-57-X
19	A Few Integer Sequences	Ibstedt, H.	1997	
20	On Smarandache's Periodic Sequences	Ibstedt, H.	1997	
21	The Smarandache Function	C. Dumitrescu, V.Seleacu	1996	1-879585-47-2 EUP

HISTORY OF THE SMARANDACHE FUNCTION

I. Bălăcenoiu and V. Seleacu
Department of Mathematics, University of Craiova
Str. Al. I. Cuza 13, Craiova 1100, Romania

1 Introduction

This function is originated from the Romanian professor Florentin Smarandache. It is defined as follows:

For any non-null integer n, $S(n) = \min\{m \mid m! \text{ is divisible by } n\}$.
So we have $S(1) = 0$, $S(2^5) = S(2^6) = S(2^7) = 8$.
If

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \dots \cdot p_t^{\alpha_t} \tag{1}$$

is the decomposition of n into primes, then

$$S(n) = \max S(p_i^{\alpha_i}) \tag{2}$$

and moreover, if $[m, n]$ is the smallest common multiple of m and n then

$$S([m, n]) = \max\{S(m), S(n)\} \tag{3}$$

Let us observe that if $\wedge = \min$, $\vee = \max$, $\underset{d}{\wedge} =$ the greatest common divizor, $\overset{d}{\vee} =$ the smallest common multiple then S is a function from the lattice $\left(\mathbf{N}^*, \underset{d}{\wedge}, \overset{d}{\vee}\right)$ into the lattice $(\mathbf{N}, \wedge, \vee)$ for which

$$S\left(\overset{d}{\bigvee_{i=\overline{1,s}}} m_i\right) = \bigvee_{i=\overline{1,s}} S(m_i) \tag{4}$$

2 The calculus of S(n)

From (2) it results that to calculate $S(n)$ is necessary and sufficient to know $S\left(p_i^{\alpha_i}\right)$. For this let p be an arbitrary prime number and

$$a_n(p) = \frac{p^n-1}{p-1} \qquad b_n(p) = p^n \tag{5}$$

If we consider the usual numerical scale

$$(p): \ b_0(p), b_1(p), \ldots, b_k(p), \ldots$$

and the generalised numerical scale

$$[p]: \ a_1(p), a_2(p), \ldots, a_n(p), \ldots$$

then from the Legendre's formula

$$\alpha! = \prod_{p_i \leq \alpha} p_i^{E_{p_i}(\alpha)} \tag{6}$$

where $E_p(\alpha) = \sum_{j\geq 1} \left[\frac{\alpha}{p^j}\right]$ it results that

$$S\left(p^{a_n(p)}\right) = b_n(p)$$

and even that: if

$$\alpha = k_\nu a_\nu(p) + k_{\nu-1}a_{\nu-1}(p) + \ldots + k_1 a_1(p) = \overline{k_\nu k_{\nu-1} \ldots k_1}_{[p]} \tag{7}$$

is the expression of α in the generalised scale $[p]$ then

$$S\left(p^\alpha\right) = k_\nu p^\nu + k_{\nu-1}p^{\nu-1} + \ldots + k_1 p \tag{8}$$

The right hand in (8) may be written as $p\left(\alpha_{[p]}\right)_{(p)}$. That is $S\left(p^\alpha\right)$ is the number obtained multiplying by p the exponent α written in the scale $[p]$ and "read" it in the scale (p). So, we have

$$S\left(p^\alpha\right) = p\left(\alpha_{[p]}\right)_{(p)} \tag{9}$$

For example to calculate $S\left(3^{100}\right)$ we write the exponent $\alpha = 100$ in the scale

$$[3]: \ 1, 4, 13, 40, 121, \ldots$$

We have $a_\nu(p) \leq p \Leftrightarrow (p^\nu - 1)/(p-1) \leq \alpha \Leftrightarrow \nu \leq \log_p((p-1)\alpha + 1)$ and so ν is the integer part of $\log_p((p-1)\alpha+1)$,

$$\nu = [\log_p((p-1)\alpha+1)]$$

For our example $\nu = [\log_3 201] = 4$. Then the first difit of $\alpha_{[3]}$ is $k_4 = [\alpha/a_4(3)] = 2$. So $100 = 2a_4(3) + 20$. For $\alpha_1 = 20$ it results $\nu_1 = [\log_3 41] = 3$ and $k_{\nu_1} = [20/a_3(3)] = 1$ so $20 = a_3(3) + 7$ and we obtain $100_{[3]} = 2a_4(3) + a_3(3) + a_2(3) + 3 = 2113_{[3]}$.
From (8) it results $S(3^{100}) = 3(2113)_{(3)} = 207$.
Indeed, from the Legendre's formula it results that the exponent of the prime p in the decomposition of $\alpha!$ is $\sum\limits_{j\geq 1}\left[\frac{\alpha}{p^j}\right]$, so the exponent of 3 in the decomposition of 207! is $\sum\limits_{j\geq 1}\left[\frac{207}{3^j}\right] = 69 + 23 + 7 + 2 = 101$ and the exponent of 3 in the decomposition of 206! is 99.
Let us observe that, as it is shown in [1], the calculus in the generalised scale $[p]$ is essentially different from the calculus in the standard scale (p), because

$$a_{n+1}(p) = pa_n(p) + 1 \text{ and } b_{n+1}(p) = pb_n(p)$$

Other formulae for the calculus of $S(p^\alpha)$ have been proved in [2] and [3].
If we note $S_p(\alpha) = S(p^\alpha)$ then it results [2] that

$$S_p(\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{10}$$

where $\sigma_{[p]}(\alpha)$ is the sum of the digits of α written in the scale $[p]$

$$\sigma_{[p]}(\alpha) = k_\nu + k_{\nu-1} + \cdots + k_1$$

and also

$$S_p(\alpha) = \frac{(p-1)^2}{p}(E_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha)$$

where $\sigma_{(p)}(\alpha)$ is the sum of digits of α written in the scale (p), or

$$S_p(\alpha) = p\left(\alpha - \left[\frac{\alpha}{p}\right] + \left[\frac{\sigma_{[p]}(\alpha)}{p}\right]\right)$$

As a direct application of the equalities (2) and (8) in [16] is solved the following problem:

"Which are the numbers with the factorial ending in 1000 zeros ?"

The solution is
$S(10^{1000}) = S(2^{1000}5^{1000}) = \max\{S(2^{1000}), S(5^{1000})\} =$
$= \max\left\{2\left(1000_{[2]}\right)_{(2)}, 5\left(1000_{[5]}\right)_{(5)}\right\} = 4005$. 4005 is the smallest natural number with the asked propriety.
4006, 4007, 4008, and 4009 verify the proprety but 4010 does not, because $4010! = 4009! \cdot 4010$ has 1001 zeros.
In [11] it presents an another calculus formula of $S(n)$:

$$S(n) = n + 1 - \left[\sum_{k=1}^{n} n^{-\left(n \sin\left(k!\frac{\pi}{n}\right)\right)^2}\right]$$

3 Solved and unsolved problems concerning the Smarandache Function

In [16] there are proposed many problems on the Smarandache Function.
M. Mudge in [12] discuses some of these problems. Many of them are unsolved until now. For example:
Problem (i) : Investigate those sets of consecutive integers $i, i+1, i+2, \ldots, i+x$ for which S generates a monotonic increasing (or indeed monotonic decreasing) sequence. (Note: For $1, 2, 3, 4, 5$, S generates the monotonic increasing sequence $0, 2, 3, 4, 5$).
Problem (ii) : Find the smallest integer k for which it is true that for all n less than some given n_0 at least one of $S(n)$, $S(n+1)$, $\ldots$, $S(n-k+1)$ is
(A) a perfect square
(B) a divisor of k^n
(C) a factorial of a positive integer
Conjecture what happens to k as n_0 tends to infinity.
Problem (iii) : Construct prime numbers of the form $\overline{S(n)S(n+1)\ldots S(n+k)}$.
For example $\overline{S(2)S(3)} = 23$ is prime, and $\overline{S(14)S(15)S(16)S(17)} = 75617$ also prime.
The first order forward finite differences of the Smarandache function are defined thus:
$D_s(x) = |S(x+1) - S(x)|$
$D_s^{(k)}(x) = D(D(\ldots \text{ k times } D_s(x)\ldots)$
Problem (iv) : Investigate the conjecture that $D_s^{(k)}(1) = 1$ or 0 for all k

greater than or equal to 2.

J. Duncan in [7] has proved that for the first 32000 natural numbers the conjecture is true.

J. Rodriguez in [14] poses the question than if it is possible to construct an increasing sequence of any (finite) length whose Smarandache values are strictly decreasing. P. Gronas in [9] and K. Khan in [10] give different solution to this question.

T. Yau in [17] ask the question that:

For any triplets of consecutive positive integers, do the values of S satisfy the Fibonacci relationship $S(n) + S(n+1) = S(n+2)$?

Checking the first 1200 positive integers the author founds just two triplets for which this holds:

$S(9) + S(10) = S(11)\,, \qquad S(119) + S(120) = S(121)\,.$

That is $S(11-2) + S(11-1) = S(11)$ and $S(11^2-2) + S(11^2-1) = S(11^2)$ but we observe that $S(11^3-2) + S(11^3-1) \neq S(11^3)\,.$

More recently Ch. Ashbacher has anounced that for n between 1200 and 1000000 there exists the following triplets satisfying the Fibonacci relationship:

$S(4900) + S(4901) = S(44902)$; $\quad S(26243) + S(26244) = S(26245)$;
$S(32110) + S(32111) = S(32112)$; $\quad S(64008) + S(64009) = S(64010)$;
$S(368138) + S(368139) = S(368140)$; $\quad S(415662) + S(415663) = S(415664)$;
but it is not known if there exists an infinity family of solutions.

The function $C_s\colon \mathbf{N}^* \mapsto \mathbf{Q}\,,\ C_s(n) = \frac{1}{n}\left(S(1) + S(2) + \cdots + S(n)\right)$ is the sum of Cesaro concerning the function S.

Problem (v) : Is there $\sum\limits_{n\geq 1} C_s^{-1}(n)$ a convergent series? Find the smallest k for which $\left(\underbrace{C_s \circ C_s \circ \cdots \circ C_s}_{k \text{ times}}\right)(m) \geq n\,.$

Problem (vi) : Study the function $S_{\min}^{-1}\colon \mathbf{N}\setminus\{1\} \mapsto \mathbf{N}\,,\ S_{\min}^{-1}(n) = \min S^{-1}(n)\,,$ where $S^{-1}(n) = \{m \in \mathbf{N} | S(m) = n\}\,.$

M. Costewitz in [6] has investigated the problem to find the cardinal of $S^{-1}(n)\,.$

In [2] it is shown that if for n we consider the standard decomposition (1) and $q_1 < q_2 < \cdots < q_s < n$ are the primes so that $p_i \neq q_j$, $i = \overline{1,t}$, $j = \overline{1,s}$, then if we note $e_i = E_{p_i}(n)$, $f_k = E_{q_k}(n)$ and $\hat{n} = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$, $\hat{n}_0 = \hat{n}/n$,

$q = q_1^{f_1} q_2^{f_2} \cdots q_s^{f_s}$, it result

$$\text{card } S^{-1}(n) = (d(\hat{n}) - d(\hat{n}_0))\, d(q) \tag{11}$$

where $d(r)$ is the number of divisors of r.
The generating function $F_S : \mathbf{N}^* \mapsto \mathbf{N}$ associated to S is defined by $F_S(n) = \sum_{d/n} S(d)$. For example $F_S(18) = S(1) + S(2) + S(3) + S(6) + S(9) + S(18) = 20$.
P. Gronas in [8] has proved that the solution of the diophantine equation $F_S(n) = n$ have the solution $n \in \{9, 16, 24\}$ or n prime.
In [11] is investigated the generating function for $n = p^\alpha$. It is shown that

$$F_S(p^\alpha) = (p-1)\frac{\alpha(\alpha+1)}{2} + \sum_{j=1}^{\alpha} \sigma_{[p]}(j) \tag{12}$$

and it is given an algorithm to calculate the sum in the right hand of (12). Also it is proved that $F_S(p_1 p_2 \cdots p_t) = \sum_{i=1}^{t} 2^{i-1} p_i$. Diophantine equations are given in [14] (see also [12]).
We mentione the followings:
(a) $S(x) = S(x+1)$ conjectured to have no solution
(b) $S(mx+n) = x$
(c) $S(mx+n) = m + nx$
(d) $S(mx+n) = x!$
(e) $S(x^m) = x^n$
(f) $S(x) + y = x + S(y)$, x and y not prime
(g) $S(x+y) = S(x) + S(y)$
(h) $S(x+y) = S(x)S(y)$
(i) $S(xy) = S(x)S(y)$
In [1] it is shown that the equation (f) has as solution every pair of composite numbers $x = p(1+q)$, $y = q(1+p)$, where p and q are consecutive primes, and that the equation (i) has no solutions $x, y > 1$.
Smarandache Function Journal, edited at the Department of Mathematics from the University of Craiova, Romania and published by Number Theory Publishing Co, Glendale, Arizona, USA, is a journal devoted to the study of Smarandache function. It publishes original material as well as reprints some that has appeared elsewhere. Manuscripts concerning new results, including computer generated are actively solicited.

4 Generalizations of the Smarandache Function

In [4] are given three generalizations of the Smarandache Function, namely the Smarandache functions of the first kind are the functions $S_n : \mathbf{N}^* \mapsto \mathbf{N}^*$ defined as follows:
(i) if $n = u^i$ ($u = 1$ or $u = p$, prime number) then $S_n(a)$ is the smallest positive integer k with the property that $k!$ is a multiple of n^a.
(ii) if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ then $S_n(a) = \max\limits_{1 \le j \le t} S_{p_j^{\alpha_j}}(a)$.
If $n = p$ then S_n is the function S_p defined by F. Smarandache in [15] ($S_p(a)$ is the smallest positive integer k such that $k!$ is divisible by p^n).
The Smarandache function of the second kind $S^k : \mathbf{N}^* \mapsto \mathbf{N}^*$ are defined by $S^k(n) = S_n(k)$, $k \in \mathbf{N}^*$.
For $k = 1$, the function S^k is the Smarandache function, with the modification that $S(1) = 1$.
If (a): $1 = a_1, a_2, \ldots, a_n, \ldots$
(b): $1 = b_1, b_2, \ldots, b_n, \ldots$
are two sequences with the property that

$$a_{kn} = a_k a_n \quad ; \quad b_{kn} = b_k b_n$$

Let $f_a^b : \mathbf{N}^* \mapsto \mathbf{N}^*$ be the function defined by $f_a^b(n) = S_{a_n}(b_n)$, (S_{a_n} is the Smarandache function of the first kind).
It is easy to see that:
(i) if $a_n = 1$ and $b_n = n$ for every $n \in \mathbf{N}^*$, then $f_a^b = S_1$.
(ii) if $a_n = n$ and $b_n = 1$ for every $n \in \mathbf{N}^*$, then $f_a^b = S^1$.
The Smarandache functions the third kind are functions $S_a^b = f_a^b$ in the case that the sequences (a) and (b) are different from those concerned in the situations (i) and (ii) from above.
In [4] it is proved that

$$S_n(a+b) \le S_n(a) + S_n(b) \le S_n(a)S_n(b) \text{ for } n > 1$$

$$\max\left\{S^k(a), S^k(b)\right\} \le S^k(ab) \le S^k(a) + S^k(b) \text{ for every } a, b \in \mathbf{N}^*$$

$$\max\left\{f_a^b(k), f_a^b(n)\right\} \le f_a^b(kn) \le b_n f_a^b(k) + b_k f_a^b(n)$$

so, for $a_n = b_n = n$ it results

$$\max\left\{S_k(k), S_n(n)\right\} \le S_{kn}(kn) \le nS_k(k) + kS_n(n) \text{ for every } k, n \in \mathbf{N}^*.$$

This relation is equivalent with the following relation written by means of the Smarandache function:

$$\max\left\{S(k^k), S(n^n)\right\} \le S\left((kn)^{kn}\right) \le nS(k^k) + kS(n^n)$$

In [5] it is presents an other generalization of the Smarandache function. Let $\mathcal{M} = \{S_m(n) | n, m \in \mathrm{N}^*\}$, let $A, B \in \mathcal{P}(\mathrm{N}^*) \setminus \emptyset$ and $a = \min A$, $b = \min B$, $a^* = \max A$, $b^* = \max B$. The set I is the set of the functions $I_A^B : \mathrm{N}^* \mapsto \mathcal{M}$ with

$$I_A^B(n) = \begin{cases} S_a(b) & , \quad \text{if } n < \max\{a, b\} \\ S_{a_k}(b_k) & , \quad \text{if } \max\{a, b\} \le n \le \max\{a^*, b^*\} \\ & \quad \text{where} \\ & \quad a_k = \max\limits_i \{a_i \in A | a_i \le n\} \\ & \quad b_k = \max\limits_j \{b_j \in B | b_j \le n\} \\ S_{a^*}(b^*) & , \quad \text{if } n > \max\{a^*, b^*\} \end{cases}$$

Let the rule $\top : I \times I \mapsto I$, $I_A^B \top I_C^D = I_{A \cup C}^{B \cup D}$ and the partial order relation $\rho \subset I \times I$, $I_A^B \rho I_C^D \Leftrightarrow A \subset C$ and $B \subset D$.
It is easy to see that $(I, \top, \rho)$ is a semilattice.
The elements u, $v \in I$ are ρ-strictly preceded by w if:
(i) $w \,\rho\, u$ and $w \,\rho\, v$
(ii) $\forall x \in I \setminus \{w\}$ so that $x \,\rho\, u$ and $x \,\rho\, v \Rightarrow x \,\rho\, w$.
Let $I^{\#} = \{(u, v) \in I \times I | u, v$ are ρ-strictly preceded $\}$, the rule $\perp : I^{\#} \mapsto I$, $I_A^B \perp I_C^D = I_{A \cap C}^{B \cap D}$ and the order partial relation r, $I_A^B r I_C^D \Leftrightarrow I_C^D \rho I_A^B$. Then the structure $(I^{\#}, \perp, r)$ is called the return of semilattice $(I, \top, \rho)$.

References

[1] M. Andrei, C. Dumitrescu, V. Seleacu, L. Tuţescu, Şt. Zanfir : Some remarks on the Smarandache Function (Smarandache Function J. Vol. 4-5, No. 1 (1994), 1-5)

[2] M. Andrei, C. Dumitrescu, V. Seleacu, L. Tuţescu, Şt. Zanfir : La fonction de Smarandache, une nouvelle fonction dans la théorie des nombres (Congrès International H. Poincaré, May 14-18, 1994, Nancy, France)

[3] M. Andrei, I. Bălăcenoiu, C. Dumitrescu, E. Rădescu, N. Rădescu, V. Seleacu : A Linear Combination with the Smarandache Function to obtain the Identity (The 26^{th} Annual Iranian Mathematics Conference, Shahid Bahonar University of Kerman, March 28-31, 1995)

[4] I. Bălăcenoiu : Smarandache Numerical Function (Smarandache Function J. Vol. 4-5, No. 1, (1994), 6-13)

[5] I. Bălăcenoiu : The semilattice with consistent return (Smarandache Notions Journal Vol. 8, 1997)

[6] M. Costewitz : Généralisation du problème 1075 (Smarandache Function J. Vol. 4-5, No. 1, (1994), 43)

[7] J. Duncan : On the conjecture $D_s^{(k)}(1) = 1$ or 0 for $k \geq 2$ (Smarandache Function J. Vol. 2-3, No. 1, (1993), 17-18)

[8] P. Gronas : The solution of the diophantine equation $\sigma_\eta(n) = n$ (Smarandache Function J. Vol. 4-5, No. 1, (1994), 14-16)

[9] P. Gronas : Letter to the Editor (Math. Spectrum, Vol. 27, No. 1, (1994-1995), 21)

[10] K. Khan : Letter to the Editor (Math. Spectrum, Vol. 27, No. 1, (1994-1995), 20-21)

[11] Maohua Le : A formula of the Smarandache Function (Smarandache Notions Journal Vol. 10, (to appear))

[12] M. Mudge : The Smarandache Function toghether with a sample of the Infinity of Unsolved Problems associated with it (Personal Computer World No. 112, (1992), 420)

[13] E. Rădescu, N. Rădescu, C. Dumitrescu : On the Sumatory Function associated to the Smarandache Function (Smarandache Function J. Vol. 4-5, No. 1, (1994), 17-21)

[14] J. Rodriguez : Problem 26.8 (Math. Spectrum, Vol. 26, No. 3, (1994), 91)

[15] F. Smarandache : A function in the Number Theory (An. Univ. Timişoara, Ser. Şt. Mat. Vol. XXVIII, fasc. 1, (1980), 79-88)

[16] F. Smarandache : An Infinity of Unsolved Problems Concerning a Function in the Number Theory (Smarandache Function J. Vol. 1, No. 1, (1990), 12-54)

[17] T. Yau : The Smarandache Function (Math. Spectrum, Vol. 26, No. 3, (1993-1994), 85)

OTHER SMARANDACHE TYPE FUNCTIONS

by J. Castillo
I40 & Window Rock Rd.
Lupton, Box 199, AZ 86508, USA

1) Let f: N ---> N be a strictly increasing function and x an element in N. Then:

   a) Inferior Smarandache f-Part of x,

      ISf(x) is the smallest k such that f(k) <= x < f(k+1).

   b) Superior Smarandache f-Part of x,

      SSf(x) is the smallest k such that f(k) < x <= f(k+1).

Particular Cases:

a) Inferior Smarandache Prime Part:
For any positive real number n one defines ISp(n) as the largest prime number less than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):
2,3,3,5,5,7,7,7,7,11,11,13,13,13,13,17,17,19,19,19,19,23,23.

b) Superior Smarandache Prime Part:
For any positive real number n one defines SSp(n) as the smallest prime number greater than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):
2,2,2,3,5,5,7,7,11,11,11,11,13,13,17,17,17,17,19,19,23,23,23.

c) Inferior Smarandache Square Part:
For any positive real number n one defines ISs(n) as the largest square less than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):
0,1,1,1,4,4,4,4,4,9,9,9,9,9,9,9,16,16,16,16,16,16,16,16,16,25,25.

b) Superior Smarandache Square Part:
For any positive real number n one defines SSs(n) as the smallest square greater than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):

0,1,4,4,4,9,9,9,9,9,16,16,16,16,16,16,16,25,25,25,25,25,25,25,25,25,36.

d) Inferior Smarandache Cubic Part:
For any positive real number n one defines ISc(n) as the largest cube less than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):
0,1,1,1,1,1,1,1,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,27,27,27,27.

e) Superior Smarandache Cube Part:
For any positive real number n one defines SSs(n) as the smallest cube greater than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):

0,1,8,8,8,8,8,8,8,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27,27.

f) Inferior Smarandache Factorial Part:
For any positive real number n one defines ISf(n) as the largest factorial less than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):

1,2,2,2,2,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,24,24,24,24,24,24,24.

g) Superior Smarandache Factorial Part:
For any positive real number n one defines SSf(n) as the smallest factorial greater than or equal to n.
The first values of this function are (Smarandache[6] and Sloane[5]):

1,2,6,6,6,6,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,24,120.

This is a generalization of the inferior/superior integer part.

2) Let g: A ---> A be a strictly increasing function, and let "~" be a given internal law on A. Then we say that f: A ---> A is <u>smarandachely complementary with respect to the function g and the internal law "~"</u> if:
f(x) is the smallest k such that there exists a z in A so that x~k = g(z).

Particular Cases:

a) Smarandache Square Complementary Function:
f: N ---> N, f(x) = the smallest k such that xk is a perfect square.
The first values of this function are (Smarandache[6] and Sloane[5]):
1,2,3,1,5,6,7,2,1,10,11,3,14,15,1,17,2,19,5,21,22,23,6,1,26,3,7.

b) Smarandache Cubic Complementary Function:
f: N ---> N, f(x) = the smallest k such that xk is a perfect cube.
The first values of this function are (Smarandache[6] and Sloane[5]):
1,4,9,2,25,36,49,1,3,100,121,18,169,196,225,4,289,12,361,50.

More generally:

c) Smarandache m-power Complementary Function:
f: N ---> N, f(x) = the smallest k such that xk is a perfect m-power.

d) Smarandache Prime Complementary Function:
f: N ---> N, f(x) = the smallest k such that x+k is a prime.
The first values of this function are (Smarandache[6] and Sloane[5]):
1,0,0,1,0,1,0,3,2,1,0,1,0,3,2,1,0,1,0,3,2,1,0,5,4,3,2,1,0,1,0,5.

References:

[1] Castillo, Jose, "Other Smarandache Type Functions", http://www.gallup.unm.edu/~smarandache/funct2.txt

[2] Dumitrescu, C., Seleacu, V., "Some Notions and Questions in Number THeory", Xiquan Publ. Hse., Phoenix-Chicago, 1994.

[3] Popescu, Marcela, Nicolescu, Mariana, "About the Smarandache Complementary Cubic Function", <Smarandache Notions Journal>, Vol. 7, no. 1-2-3, 54-62, 1996.

[4] Popescu, Marcela, Seleacu, Vasile, "About the Smarandache Complementary Prime Function", <Smarandache Notions Journal>, Vol. 7, no. 1-2-3, 12-22, 1996.

[5] Sloane, N.J.A.S, Plouffe, S., "The Encyclopedia of Integer Sequences", online, email: superseeker@research.att.com (SUPERSEEKER by N. J. A. Sloane, S. Plouffe, B. Salvy, ATT Bell Labs, Murray Hill, NJ 07974, USA).

[6] Smarandache, Florentin, "Only Problems, not Solutions!", Xiquan Publishing House, Phoenix-Chicago, 1990, 1991, 1993; ISBN: 1-879585-00-6.
(reviewed in <Zentralblatt fur Mathematik> by P. Kiss: 11002, 744, 1992;
and in <The American Mathematical Monthly>, Aug.-Sept. 1991);

[7] "The Florentin Smarandache papers" Special Collection, Arizona State University, Hayden Library, Tempe, Box 871006, AZ 85287-1006, USA;
(Carol Moore & Marilyn Wurzburger: librarians).

SURVEY ON THE RESEARCH OF SMARANDACHE NOTIONS

by M. L. Perez, editor

The American CRC Press, Boca Raton, Florida, published, in December 1998, a 2000 pages "**CRC Concise Encyclopedia of Mathematics**", by Eric W. Weisstein, ISBN 0-8493-9640-9, internationally distributed.

Among the entries included in this prestigious encyclopedia there also are the following:

- "Smarandache functions"

[i.e., Pseudosmarandache Function (p. 1459), Smarandache Ceil Function (p. 1659), Smarandache Function (p. 1660) - the most known, Smarandache-Kurepa Function (p. 1661), Smarandache Near-to-Primordial (p. 1661)], Smarandache-Wagstaff Function (p. 1663)]

- "Smarandache sequences" [41 such sequences are listed (pp. 1661-1663), in addition of 7 other Smarandache concatenated sequences (pp. 310-311))
- "Smarandache constants" [11 such constants are listed (pp. 1659-1660)]
- "Smarandache paradox" (p. 1661).

Five large pages from the above encyclopedia are dedicated to these notions.

Other contributors to the Smarandache Notions are cited as well in this wonderful mathematical treasure: C. Ashbacher, A. Begay, M. Bencze, J. Brown, E. Burton, I. Cojocaru, S. Cojocaru, J. Castillo, C. Dumitrescu, Steven Finch, E. Hamel, F. Iacobescu, H. Ibstedt, K. Kashihara, H. Marimutha, M. Mudge, I. M. Radu, J. Sandor, V. Seleacu, N. J. A. Sloane, S. Smith, Ralf W. Stephan, L. Tutescu, David W. Wilson, E. W. Weisstein, etc.

Professor Eric W. Weisstein from the University of Virginia has extended more results on Smarandache sequences, such as:

- The Smarandache Concatenated Odd Sequence:

1, 13, 135, 1357, 13579, 1357911, 135791113, 13579111315, ...
(Sloane's A019519) contains another prime term:
SCOS(2570) = 13579111315...51375139, which has 9725 digits!
This is the largest consecutive odd number sequence prime ever found.

Conjecture 1: There is a finite number of primes in this sequence.

- The Smarandache Concatenated Prime Sequence:

2, 23, 235, 2357, 235711, 23571113, 2357111317, ...
(Sloane's A019518) is prime for terms 1, 2, 4, 128, 174, 342, 435, 1429, ... (Sloane's A046035) with no other less than 1960.

Conjecture 2: There is a finite number of primes in this sequence.

- The Smarandache Concatenated Square Sequence:

1, 14, 149, 14916, 1491625, 149162536, 14916253649, ...
(Sloane's A019521) contains a prime only 149 (the third term) in the first 1828 terms.

Conjecture 3: There is only a prime in this sequence.

- The Smarandache Concatenated Cubic Sequence:
1, 18, 1827, 182764, 182764125, 182764125216, ...
(Sloane's A019522) contains no prime in the first 1356 terms.
Conjecture 4: There is no prime in this sequence.

David W. Wilson (wilson@cabletron.com) proved that
- The Smarandache Permutation Sequence:
12, 1342, 135642, 13578642, 13579108642, 13579112108642, 1357911131412108642, ...
has no perfect power in its terms.

Proof:
Their last digits should be:
either 2 for exponents of the form 4k+1,
either 8 for exponents of the form 4k+3, where $k \geq 0$.
12 is not a perfect power. All remaining elements are congruent to 2 (mod 4), and are therefore not a perfect power, either. QED.
- The Smarandache Binary Sieve (Item 29 in http;//www.gallup.unm.edu/~smarandache/SNAQINT.txt):
1,3,5,9,11,13,17,21,25,27,29,33,35,37,43,49,51,53,57,59,65,67,69,73,75,77,81,85,89,91,97,101,107,109,113,115,117,121,123,129,131,133,137,139,145,149,...

(Starting to count on the natural numbers set at any step from 1:
- delete every 2-nd numbers
- delete, from the remaining ones, every 4-th numbers
... and so on: delete, from the remaining ones, every 2^k-th numbers, k = 1, 2, 3, ...)

Conjectures:
a) There are an infinity of primes that belong to this sequence;
b) There are an infinity of numbers of this sequence which are composite.

The second conjecture has been proved true by David W. Wilson:
One way to see this is to note that any sequence with positive density over the positive integers contains an infinitude of composites (the density of this sequence is
$1/2 * 3/4 * 7/8 * 15/16 * 31/32 * ... = 0.288788095086602421 27... > 0$.)
Another way to see this is to note that this sequence contains all numbers of the form $(4^k-1)/3$ for $k \geq 3$, which are all composite.

Also, in the "Bulletin of Pure and Applied Sciences", Delhi, India, Vol. 17E, No. 1, 1998 (pp. 103-114, 115-116, 117-118, 123-124) four articles present the "<u>Smarandache noneuclidean geometries</u>".

References:

[1] C. Dumitrescu, V. Seleacu, "Some Notions and Questions in NumberTheory",http;//www.gallup.unm.edu/~smarandache/SNAQINT.txt.
[2] E. W. Weisstein, E-mails to J. Castillo, March-December 1998.
[3] D. W. Wilson, E-mails to J. Castillo, Fall-Winter 1998.

Logic As Algebra, by Paul Halmos and Steven Givant, The Mathematical Association of America, Washington, D. C., 1998. 152 pp., $27.00(paper). ISBN 0-88385327-2.

It can be strongly argued that logic is the most ancient of all the mathematical sub-disciplines. When mathematics as we know it was being created so many years ago, it was necessary for the concepts of rigid analytical reasoning to be developed. Of the three earliest areas, geometry was born out of the necessity of accurately measuring land plots and large buildings and number theory was required for sophisticated counting techniques. Logic, the third area, had no "practical" godfather, other than being the foundation for rigorous reasoning in the other two. In the intervening years, so many additional areas of mathematics have been developed, with logic and logical reasoning continuing to be the fundamental building block of them all. Therefore, every mathematician should have some exposure to logic, with the simple history lesson automatically being included. This short, but excellent book fills that niche.

The title accurately sets the theme for the entire book. Algebra is nothing more than a precise notation in combination with a rigorous set of rules of behavior. When logic is approached in that way, it becomes much easier to understand and apply. This is especially necessary in the modern world where computing is so ubiquitous. Many areas of mathematics are incorporated into the computer science major, but none is more widely used than logic. Written at a level that can be comprehended by anyone in either a computer science or mathematics major, it can be used as a textbook in any course targeted at these audiences.

The topics covered are standard although the algebraic approach makes it unique. One simple chapter subheading, 'Language As An Algebra', succinctly describes the theme. Propositional calculus, Boolean algebra, lattices and predicate calculus are the main areas examined. While the treatment is short, it is thorough, providing all necessary details for a sound foundation in the subject. While the word "readable" is sometimes overused in describing books, it can be used here without hesitation.

Sometimes neglected as an area of study in their curricula, logic is an essential part of all mathematics and computer training, whether formal or informal. The authors use a relatively small number of pages to present an extensive amount of knowledge in an easily understandable way. I strongly recommend this book.

Reviewed by

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, Iowa
71603.522@compuserve.com

In Polya's Footsteps: Miscellaneous Problems and Essays, by Ross Honsberger, The Mathematical Association of America, Washington, D. C., 1997. 328 pp., $28.95(paper). ISBN 0-88385-326-4.

The greatest scientist of all time was quoted as saying that the reason that he saw further than others was that he stood on the shoulders of giants. As the title of this book suggests, there is another route, namely walking the same path as others. Given our individual differences and how we vary from day to day, even the most beaten of paths can present differing appearances. When walking through a forest, some days you may see the moss, other days the ground cover and then on others we pay particular attention to the leaves. In this collection of problems, Ross

Honsberger proves once again that he is the best at picking the high quality, sturdy building material from the large, stable, yet uninspiring stack of wood.

This is a collection of problems to build on. Many of the them were taken from those proposed and rejected from mathematics competitions, both national and international. Given the quality of these problems, those that were accepted in favor of them must have indeed been gems. It is fortunate that **Crux Mathematicorum**, a journal of the Canadian Mathematical Society, publishes problems of this type so that the rest of us may enjoy them. The range of topics is extensive, with very detailed proofs of all problems. The most striking aspect of many of them is that the approach used in the proof is "non-obvious." Which is the mathematical term for ,"now, how did they ever think of that?" Which is what makes them so charming and emphasizes how exciting mathematics is. There used to be a television game show where contestants competed by claiming that they could name a song in the fewest notes. If there was a similar contest concerning the elegance and directness of proofs, some of those in this book would provide stiff competition.

Classic works of art or music always provide enjoyment, even after many repetitions. High quality, elegant proofs of mathematical problems do the same thing to those willing to experience them. This is one book that will allow you to do that.

Reviewed by

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233

Computer Analisis of Number Sequences, by Henry Ibstedt, American Research Press, Lupton, Az., 1998. 87 pp., $9.95 (paper), ISBN 1-879585-59-6.

Playing with numbers is one activity that all mathematicians enjoy. It is considered a pleasurable occupational hazard. Finding "new" properties of numbers is a joy that cannot be accurately described, only experienced. In this book, the author presents and to some extent explores a set of problems in recreational mathematics. Nearly all of the problems originated in the mind of Florentin Smarandache, the creator of innumerable problems in many areas of mathematics. While many are somewhat contrived, they are all fun to read and think through.

For exemple, there are the three sequences of numbers formed by the repeated concatenation of the elements of a set of integers

Smarandache Odd Sequence (SOS):
1, 13, 135, 1357, 13579, 1357911, 135791113, ...

Smarandache Even Sequence (SES):
2, 24, 246, 2468, 246810, 24681012, ...

Smarandache Prime Sequence (SPS):
23, 235, 2357, 235711, 23571113, ...

where questions like the following are presented.

How many primes are there in the SOS and SPS sequences?
How many perfect powers are there in the SES sequence?

Like the large Marsenne primes, the current largest Known prime in either of these sequences is an accurate barometer of the state of current factoring capability. As no less a mathemetician as Pal Erdos has noted, it will probably never be known if there is an infinite number of primes in either the SOS or SPS sequences. However, if someone ever resolves the issue, it will no doubt be headline news in the mathematics community. Any technique powerfull enough to resolve this issue will certainly be one that can be used elsewhere.
It is just an interesting collection of problems in recreational mathematics that can be worked on just for the joy of exploration. That alone makes it well worth reading.

Reviewed by

Charles Ashbacher
Charles Ashbacher Technologies
Box 294
Hiawatha, IA 52233, USA

CRC Concise Encyclopedia of Mathematics, by Eric W. Weisstein, CRC Press, Boca Raton, FL, USA, 1998, 1969 pp., $79.95 (alk. paper), ISBN 0-8493-9640-9.

The best ever published encyclopedia of mathematics. Also very accessible and well organized, with many cross-references.

From

http://www.amazon.com/exec/obidos/ISBN=0849396409/ericstreasuretroA/,

"The CRC Concise Encyclopedia of Mathematics is a compendium of mathematical definitions, formulas, figures, tabulations, and references. Its informal style makes it accessible to a broad spectrum of readers with a diverse range of mathematical backgrounds and interests. This fascinating, useful book draws connections to other areas of mathematics and science as well as demonstrates its actual implementation providing a highly readable, distinctive text diverging from the all-too-frequent specialized jargon and dry formal exposition.

Through its thousands of explicit examples, formulas, and derivations, The CRC Concise Encyclopedia of Mathematics gives the reader a flavor of the subject without getting lost in minutiae, stimulating his or her thirst for additional information and exploration.

This book serves as handbook, dictionary, and encyclopedia extensively cross-linked and cross-referenced, not only to other related entries, but also to web sites on the Internet. Standard mathematical references, combined with a few popular ones, are also given at the end of most entries, providing a resource for more reading and exploration. In The CRC Concise Encyclopedia of Mathematics, the most useful and interesting aspects of the topic are thoroughly discussed, enhancing technical definitions."

C O N T E N T S

ARTICLES:

Micha Fleuren, Smarandache Factors and Reverse Factors 5
Henry Ibstedt, Smarandache Continued Fractions 39
Constantin Dumitrescu, Carmen Rocsoreanu, Some Connections Between the Smarandache Function and the Fibonnaci Sequence 50
Y. V. Chebrakov, V. V. Shmagin, Computative Paradoxes in Modern Data Analysis 61
Kevin Ford, The Normal Behavior of the Smarandache Function 81
J. Sándor, On Values of Arithmetical Functions at Factorials I 87
Steven R. Finch, The Average Value of the Smarandache Function 95
Sabin Tabirca, Smarandache Magic Squares 100
Sabin Tabircă, Tatiana Tabircă, Some Inequalities Concerning Smarandache's Function 105
Sebastián Martín Ruiz, Smarandache's Function Applied to Perfect Numbers 114
Florian Luca, On the Divergence of the Smarandache Harmonic Series 116
Sabin Tabircă, Tatiana Tabircă, A Generalization of Euler's Function 119
Sebastián Martín Ruiz, A Result Obtained Using Smarandache's Function 123
J. Sándor, On An Inequality for the Smarandache Function ... 125
Florian Luca, On A Series Involving S(1)·S(2)...·S(n) 128
Sebastián Martín Ruiz, A Congruence with Smarandache's Function 130
Henry Ibstedt, An Integer as a Sum of Consecutive Integers . 133

NOTES:

Mihaly Bencze, A New Inequality for the Smarandache Function 139
Maohua Le, A Formula for the Smarandache Function 140
Maohua Le, On the Diophantine Equation S(n) = n 142
Maohua Le, On Smarandache Divisor Product 144
Maohua Le, On the Smarandache n-ary Sieve 146
Maohua Le, Perfect Powers in the Smarandache Permutation Sequence 148
Maohua Le, On Smarandache Pseudo-Powwers of Third Kind 150
Maohua Le, An Improvement on the Smarandache Divisibility Theorem 152
Maohua Le, On Primes in the Smarandache Pierced Chain 154
Maohua Le, Primes in the Sequences $\{n^n+1\}_{n=1}^{\infty}$ and $\{n^n+1\}_{n=1}^{\infty}$ 156
Maohua Le, On the Smarandache Prime Additive Complement Sequence 158
Mihaly Bencze, An Inequality for the Smarandache Function .. 160
Maohua Le, On Smarandache Simple Functions 161
Charles Ashbacher, Maohua Le, On Smarandache Simple Continued

Fractions 163
Maohua Le, Notes on Primes Smarandache Progressions 165
Maohua Le, The Primes p with lg(p) = 1 167
Maohua Le, Some Solutions of the Smarandache Prime Equation 169
Charles Ashbacher, Smarandache Series Converges 171
Maohua Le, A Note on Primes in the Sequence $\{a^n+b\}_{n=1}^{\infty}$ 172
Maohua Le, The Primes in the Smarandache Symmetric Sequence 174
Maohua Le, On Smarandache General Continued Fractions 176
Maohua Le, The Lower Bound for the Smarandache Counter C(0,n!) 178
Maohua Le, On Smarandache Pseudo-Primes of Second Kind 180

MATHEMATICAL HISTORY:

Henry Ibstedt, Summary of the First International Conference on Smarandache Type Notions in Number Theory (University of Craiova, August 21-24, 1997) 181
Henry Ibstedt, Preambul to the First International Conference on Smarandache Type Notions in Number Theory (University of Craiova, August 21-24, 1997) 182
Henry Ibstedt, The Smarandache Sequence Inventory 183
Henry Ibstedt, List of Smarandache Documents 191
I. Bălăcenoiu, V. Seleacu, History of the Smarandache Function 192
J. Castillo, Other Smarandache Type Functions 202
M. L. Perez, Survey on the Research of Smarandache Notions 205

REVIEWS:

Book Reviews 207

The "Smarandache Notions" is now online at:
http://www.gallup.unm.edu/~smarandache/
and selected papers are periodically added to our web site. The authors are asked to send, together with their hard copy manuscripts, a floppy disk with HTML or ASCHII files to be put in the Internet as well.

Papers in electronic form are accepted. They can be e-mailed in Microsoft Word 7.0a (or lower) for Windows 95, WordPerfect 6.0 (or lower) for Windows 95, or text files.

Starting with Vol. 10, the "Smarandache Notions" is also printed in hard cover for a special price of $39.95.

Mr. Pål Grønås from Norway successfully defended his Master Degree (*Hovedfag*) thesis in mathematics with the title "Grøbner-genererende mengder for idealer i en spesiell type algebraer -- Smarandache-funksjonen" (Norwegian) in 1998 under the supervision of Professor Dr. Øyvind Solberg.

$29.95